Math. 224. Review problems 1

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Topics

1. **Modeling:** growth-decay (population et al), heating-cooling, free fall, mixing, chemical reactions, financing, mechanics, geometry;
   (a) **Multi-D models**: interacting species (competition-cooperation), predator-prey, oscillators
2. **Concepts:** initial value problem, general solution, uniqueness, equilibriua, phase line
3. **Methods, techniques:**
   (a) **Analytic**: separation, mulitpiers, change of variables, linear approximation
   (b) **Qualitative**: slope field, phase line, equilibria (stability and bifurcations)
   (c) **Numeric**: Euler

Sample problems

4. **Modeling.** Write differential equation models for the following systems. Classify them as DE/DS, autonomous/non-autonomous, first or higher order, linear/nonlinear, separable etc. Indicate solution methods for each one.
   (a) The logistic growth model with the growth rate α =.4, threshold capacity N = 20, initial population of 30, and different harvesting methods (fixed, variable, proportional to p...)
   (b) Oscillator (mass-spring) system of mass m, damping coefficient α, spring force with spring constant k, and nonlinear oscillator with cubic force
   (c) A mixing container with inflow rate $r_1 = 2$ gal/sec and (incoming) concentration α mg/gal; outflow rate r gal/sec; initial volume 20 gal, initial concentration 0. Discuss cases $r = 2$ and $r \neq 2$.
   (d) Financing loan at interest α and payment rate p (minimal payment rate and duration of the loan for a given p).
   (e) Chemical reactions: $A + B \overset{k_1}{\rightarrow} C$; $A + C \overset{k_2}{\rightarrow} B$
   (f) Jump from height H =3000 m, following 50 sec free fall, if friction coefficients are α_1 (free fall), α_2 (parachutte)
   (g) Predator-prey system with prey reproduction rate =.5 prey/hr; killing rate =2 prey/(prey,pred,hr); predator growth rate =.3 pred/(prey,pred,hr); predator attrition rate =.02 pred/hr. Assume furthermore, that prey are introduced in the system at a constant rate a prey/hr, while predators are hunted at a rate proportional to their population.

(h) Two competing/cooperating species that reproduce at rates $\alpha_1 = .5$, $\alpha_2 = 1.4$, have threshold capacities N_1 and N_2, assuming each species inhibits the other species' growth at a rate proportional to the product of two populations with coefficients $b_1; b_2$

5. **Solution methods**

(a) Write the general solution of the logistic model 1(a) without harvesting. Find IVP-solution $y(0) = 5$. Plot equilibria and 3 solution curves: $y(0) = 5$; $y(0) = 10$; $y(0) = 25$

(b) Solve the mixing problem 1(c) with outgoing rate $r = 2$ and the initial concentration 0. Describe what happens to concentration $\alpha(t)$ as $t \to \infty$. Same for $r = 3$

(c) Can DE $y' + 2y = \cos t$ have a periodic solution? (Hint: try $y(t) = A\cos t + B\sin t$ with undetermined coefficients A, B. Find A, B)

(d) Solve the cooling problem for two given temperature readings $T_1 = 83$, $T_2 = 72$, taken 3 hours apart. Find the time when temperature reached 65, plot solution and describe its behavior as $t \to \infty$

(e) Find the landing time for problem 1(f), if $\alpha_1 = .2[1/\sec]$, $\alpha_2 = 15[1/\sec]$

6. **Phase-line. Bifurcations.** Consider growth model $y' = y(y-2)(3-y) - a$, (constant harvest rate)

(a) Find the bifurcation values of parameter a and plot the bifurcation (fork) diagram

(b) Sketch phase-lines, direction fields and solution curves for 3 values of a above and below the bifurcation values

(c) Locate equilibria (approximately) and describe their stability-type (sink/source) in each case

(d) Discuss the qualitative change of the particular IVP-solution $y(0) = 2$ (fixed initial value) as a passes through the bifurcation values

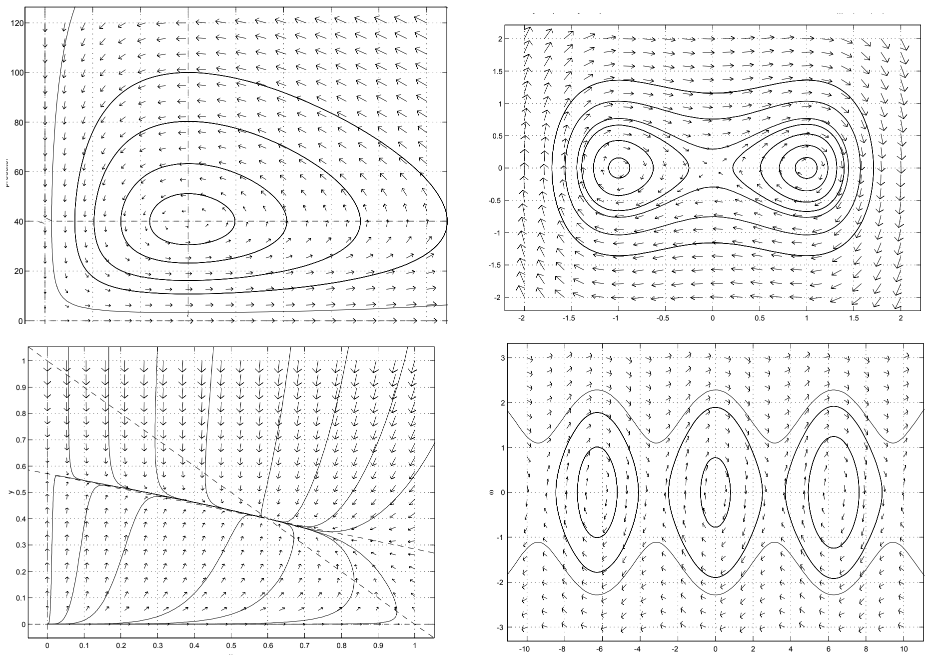
7. Do similar analysis for harbesting $-ay$

8. Discuss the effect of harvesting the logistic model $f = .3y(1 - y/20) - by$. Which values b allow to sustain the population at some level?

9. **Dynamical systems; phase-plots; equilibria**

(a) Identify each plot with a suitable dynamical system (vector-field): $F = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$ from the following list, and a suitable physical model (predator-prey; oscillator with friction; pendulum; competing species)

1) $\begin{pmatrix} y \\ -3x - .4y \end{pmatrix}$	2) $\begin{pmatrix} x(1-x-y) \\ y(4-7y-2x) \end{pmatrix}$	3) $\begin{pmatrix} x - .5xy \\ .3xy - .4y \end{pmatrix}$	4) $\begin{pmatrix} y \\ -\sin x \end{pmatrix}$



(b) Find equilibria for each of the systems and locate them on the plot

(c) Pick a particular (marked) trajectory for each one of 4 phase-plots and sketch its solution-curves $x(t)$ (solid) and $y(t)$ (dashed). Describe in words what happens to your solutions at large time t

(d) Write analytic solution of a linear oscillator model of mass $m = 2$ and spring constant $k = 5$ (no damping) with the initial state $x(0) = 3, v(0) = 0$. Plot its trajectory and solution curves $x(t), v(t)$.