Math. 224: Test II, Apr.17, 2008

Print Name: ______________________________

1. Systems and models:

a) Migrating populations $\{u,v,w\}$ between 3 areas ($i=1,2,3$), with the following migration rates a_{ij} ($j \to i$) per unit time: $a_{12}=a_{23}=2; a_{31}=1.5; a_{21}=a_{32}=.5; a_{13}=2$. Write the system in the matrix-vector form, compute matrix A. Explain how you expect solutions to behave at large time, and what happens to the total population. Outline solution method (**don't solve**!)

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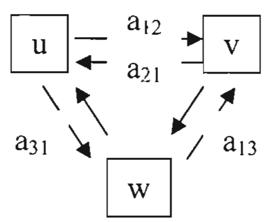

$$\begin{cases} \dot u = -(a_{21}+a_{31})u + a_{12}v + a_{13}w \\ \dot v = a_{21}u - (a_{12}+a_{32})v + a_{23}w \\ \dot w = a_{31}u + a_{32}v - (a_{13}+a_{23})w \end{cases}$$

$$A = \begin{bmatrix} -(.5+1.5) & 2 & 2 \\ .5 & -(2.5) & 2 \\ 1.5 & .5 & -4 \end{bmatrix};$$

Total: $u+v+w = N$ - const

$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} \to$ equilibrium

Solution method – E-Value problem

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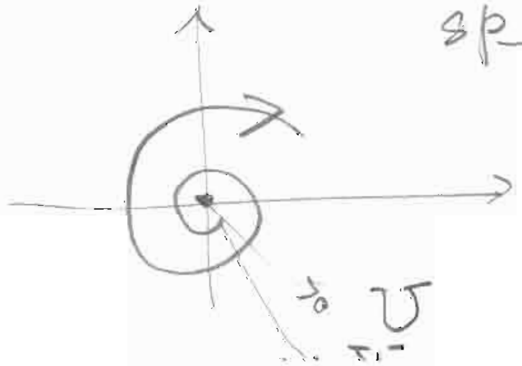
b) A real 2×2 matrix A has complex eigendata: $\lambda = 1+2i; X=(1-i,-1+2i)$. Write real general solution (real functions), sketch phase plane and solution curves $x(t), y(t)$, describe equilibria type. (**Extra**: find matrix A)

Complex: $Y_c(t) = e^{\lambda t}X = e^t(\cos 2t + i\sin 2t)\left[\binom{1}{-1} + i\binom{-1}{2}\right]$ (U, V)

$\operatorname{Re} Y_c = e^t\left[\cos 2t\binom{1}{-1} - \sin 2t\binom{-1}{2}\right]$ ← Real pair

$\operatorname{Im} Y_c = e^t\left[\cos 2t\binom{-1}{2} + \sin 2t\binom{1}{-1}\right]$ ←

sp. source



Problem	Score
1(25)	
2(25)	
3(25)	
4(25)	
Total	

c) Write Newton equations of motion (2nd order DE and 2D system) for nonlinear oscillator with force function $f(x) = x^3 - x$. Formulate energy conservation law (find potential energy) for this system. Explain the effect of friction on energy.

$$m\ddot{x} = x^3 - x \iff \begin{cases} \dot{x} = v \\ \dot{v} = \frac{1}{m}(x^3 - x) \end{cases}$$

Pot. energy: $U(x) = \frac{x^2}{2} - \frac{x^4}{4}$

Total: $E = \frac{m\dot{x}^2}{2} + U(x) - \text{const} \iff \frac{dE}{dt} = 0$

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Friction $= -\alpha\dot{x}$ dissipates energy

$$\frac{dE}{dt} = -\alpha\dot{x}^2 \le 0$$

d) Predator-prey with logistic prey: $\begin{array}{l} \dot{x} = x(1 - ax - y) \\ \dot{y} = y(x - 2) \end{array}$. Explain the meaning of variables and coefficients (*a*-?). Sketch phase-plane (null-clines, equilibria) in one of 2 cases: *a*=*1/4* or *a*=*1* (**Extra**-credit for both)

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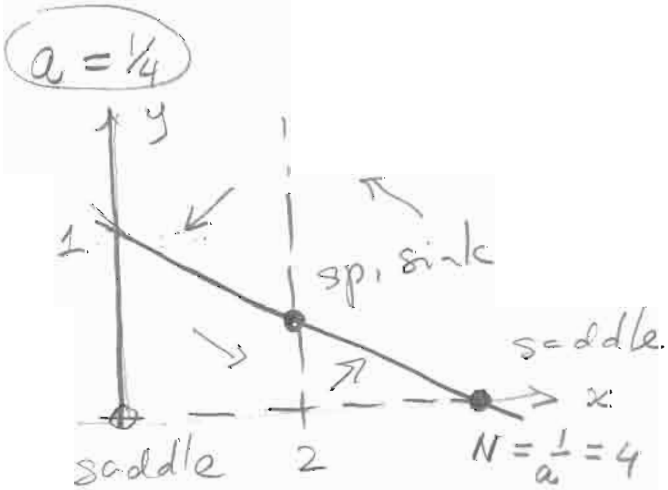
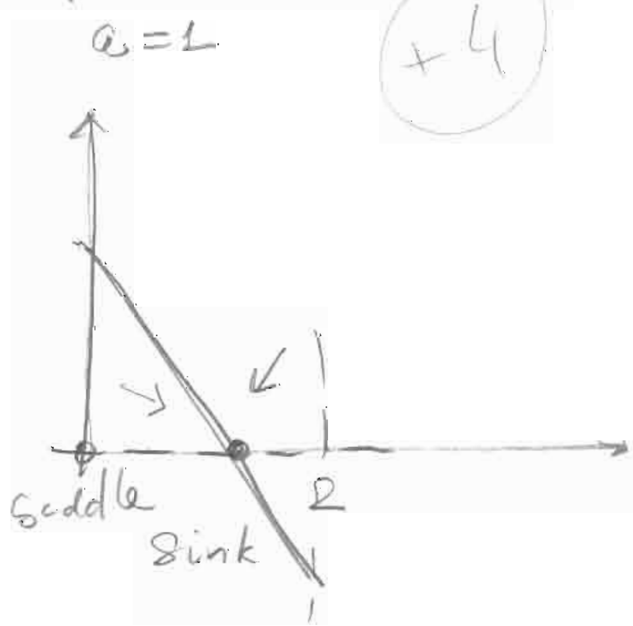
x - prey, y - pred. | a = 1/c.c. prey, 1 = killing rate / pred. x prey, 2 = attrition of pred.

+4





2. **Linear systems and bifurcation problem.** Take matrix family: $A(b)=\begin{bmatrix} b & 2b \\ 1 & 1 \end{bmatrix}$.

a) For $b=1$, solve the eigenvalue problem and write general solution $\frac{dY}{dt}=A\cdot Y$. Sketch phase-plane trajectories and typical solution curves $x(t), y(t)$. Describe equilibrium type.

$A=\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad p=\lambda^2-2\lambda-1$

λ	$1-\sqrt{2}$	$1+\sqrt{2}$
X	$\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$	$\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$

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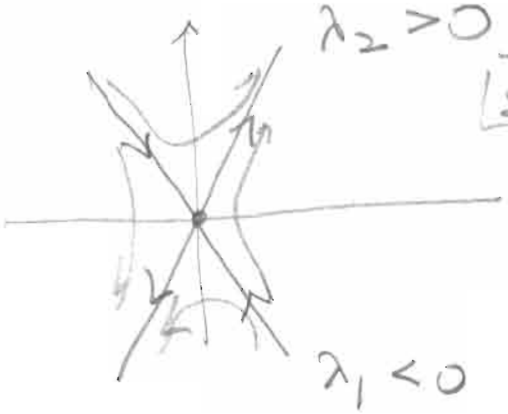
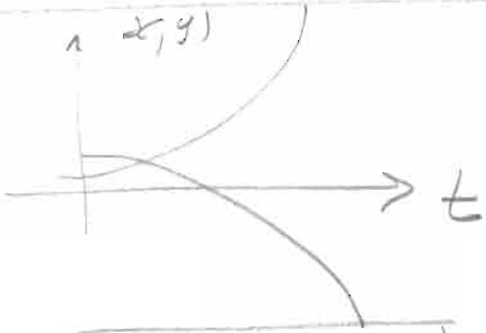


saddle

$$Y(t)=c_1e^{\lambda_1 t}X_1+c_2e^{\lambda_2 t}X_2$$



b) Same as part (a) for $b=-1$.

$A=\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$ $\quad p=\lambda^2+1$

$\lambda=\pm i$
$X=\begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

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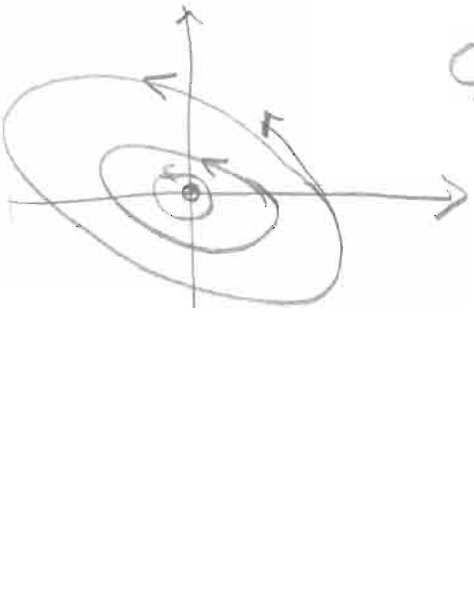

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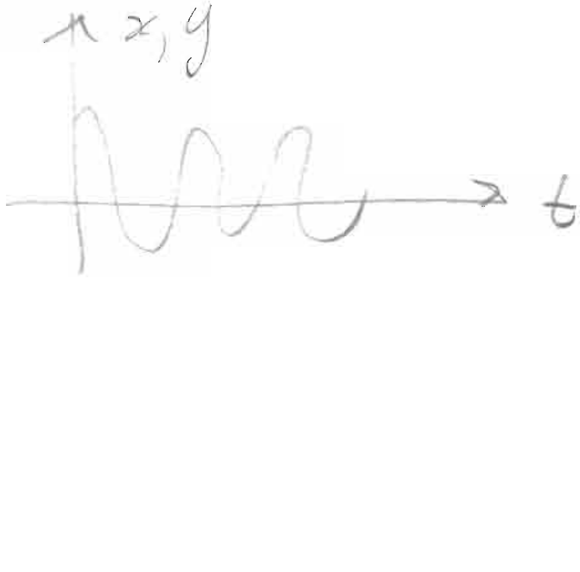

c) Solve bifurcation problem for matrix family $\{A(b)\}$: (i) sketch bifurcation diagram in the trace-determinant plane; (ii) find bifurcation values $\{b_j^*\}$, and describe equilibria types and transitions between all b-ranges. Sketch schematic phase-plot for each range.

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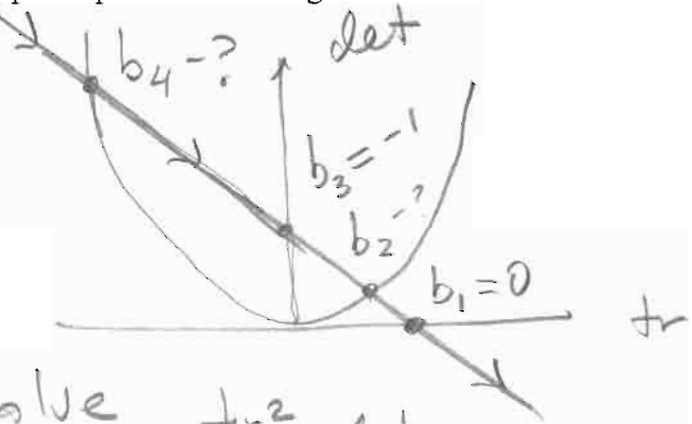
$$\operatorname{tr} A = b+1$$
$$\det A = -b$$

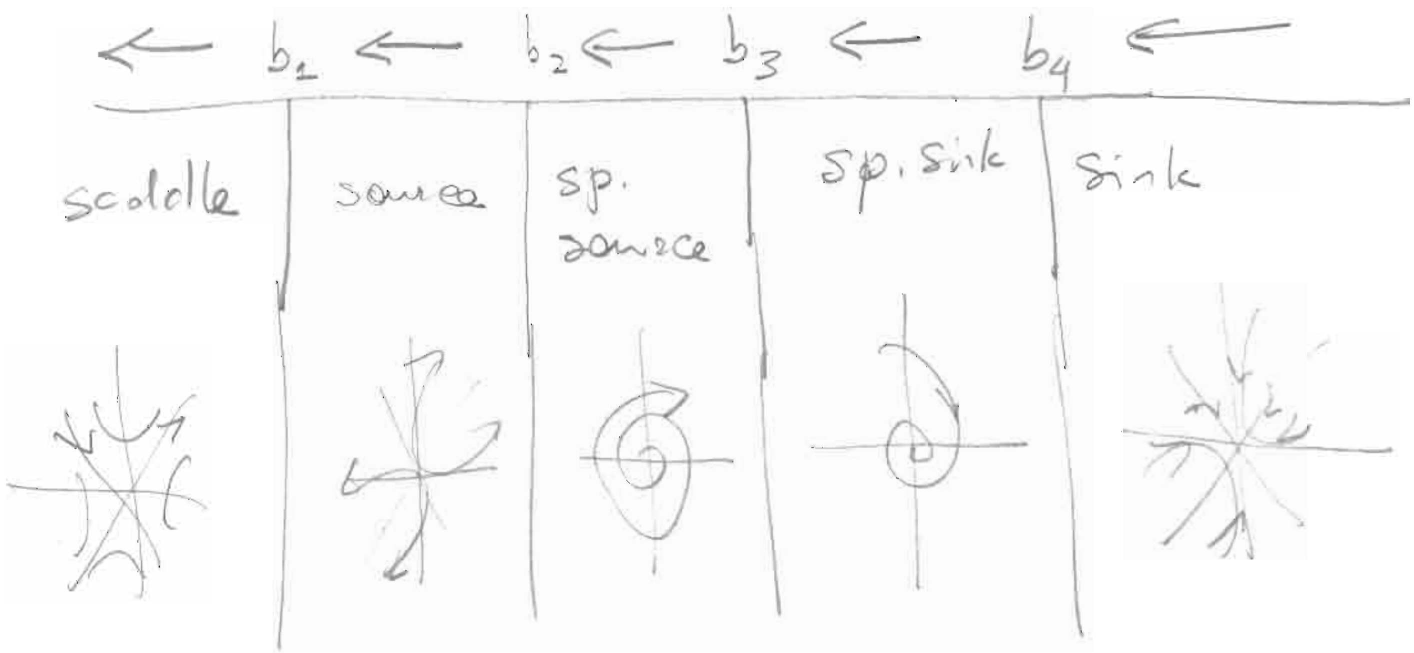


To find $b_{2,4}$ solve $\frac{tr^2}{4} = \det$

$$\frac{(b+1)^2}{4} + b = 0 \implies b^2 + 6b + 1 = 0$$

$$b_{2,4} = -3 \pm \sqrt{2}$$

$b_2 = -3+\sqrt{2}$
$b_4 = -3-\sqrt{2}$



3. **Characteristic polynomial method**. Find particular solution, homogeneous solutions, general and IVP solution for each of the following problems. (Hint: for periodic sources $\cos\omega t; \sin\omega t$ you can use complex exponential solutions: $\mathrm{Re}\left(\frac{e^{i\omega t}}{p(i\omega)}\right), \mathrm{Im}(...)$).

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DE	Char. Polynomial $p(\lambda)$	Char. roots $\{\lambda_i\}$ and homogeneous solution	Particular (response) solution
a) $y'+2y=1+\cos t$;	$\lambda+2$	$\lambda_1=-2$, $y_h=e^{-2t}$	$\frac{1}{2}+\frac{2\cos t+\sin t}{5}$
b) $y''+4y=\cos\beta t$;	λ^2+4	$\lambda_{1,2}=\pm 2i$, $\{\cos 2t, \sin 2t\}$	$\frac{\cos\beta t}{4-\beta^2}$
c) $y''+2y'+4y=\sin\beta t$	$\lambda^2+2\lambda+4$	$\lambda=-1\pm i\sqrt{3}$, $(e^{-t}\cos\sqrt{3}t, e^{-t}\sin\sqrt{3}t)$	$\frac{(4-\beta^2)\sin\beta t-2\beta\cos\beta t}{(4-\beta^2)^2+4\beta^2}$

a) Solve IVP $y(0)=-1$ for problem (a). Sketch y(t)

$y=y_p+ce^{-2t}\Big|_{t=0}=\left(\frac{1}{2}+\frac{2}{5}\right)+c=-1$

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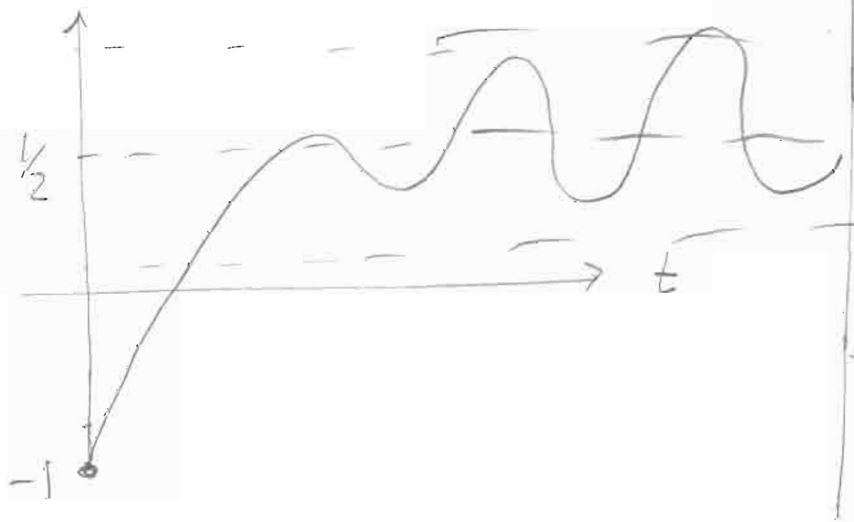
$c=-1.9$



Particular

a) $y_p=\frac{1}{2}+\mathrm{Re}\left[\frac{e^{it}}{p(i)}\right]=$

$=\frac{1}{2}+\mathrm{Re}\left[\frac{(\cos t+i\sin t)(2-i)}{5}\right]$

$=\frac{1}{2}+\frac{2\cos t+\sin t}{5}$

b) $y_p=\mathrm{Re}\left[\frac{e^{i\beta t}}{p(i\beta)}\right]=\mathrm{Re}\left[\frac{e^{i\beta t}}{4-\beta^2}\right]$

$=\frac{\cos\beta t}{4-\beta^2}$

c) $y_p=\mathrm{Im}\left[\frac{e^{i\beta t}}{(4-\beta^2)+2i\beta}\right]=$

$=\mathrm{Im}\left[\frac{e^{i\beta t}[(4-\beta^2)-2i\beta]}{(4-\beta^2)^2+4\beta^2}\right]=$

$\frac{(4-\beta^2)\sin\beta t-2\beta\cos\beta t}{(4-\beta^2)^2+4\beta^2}$

as

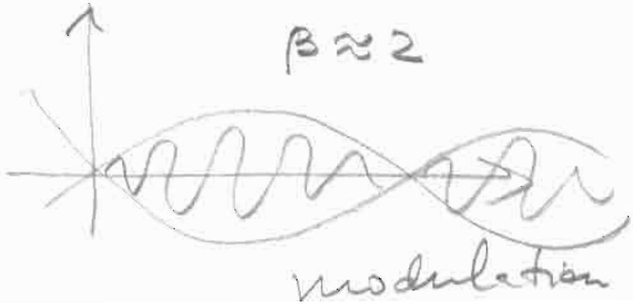
b) Explain (b) "oscillator" problem (meaning of y, coefficients et al). Solve IVP: $y(0) = y'(0) = 0$ for (b). Sketch solutions and explain what happens as β approaches 2 (modulation, resonance)

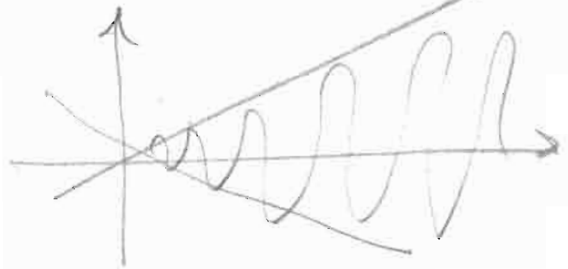
$\frac{k}{m} = 4$, no damping; y - position

$$y = \frac{\cos\beta t}{4-\beta^2} + C_1 \cos 2t + C_2 \sin 2t \Rightarrow \begin{cases} y(0) = \frac{1}{4-\beta^2} + C_1 = 0 \\ \dot{y}(0) = 2C_2 = 0 \end{cases}$$

7 $\Rightarrow$ $$y(t) = \frac{\cos\beta t - \cos 2t}{4-\beta^2}$$

$\beta \approx 2$





c) Explain (c) as an "oscillator" problem (meaning of y et al). Write particular (response) solution for in the "amplitude-phase shift" form, sketch the response amplitude $A(\beta)$. {Extra: find maximal response amplitude)

y - position, $4 = \frac{k}{m}$; 2 = friction

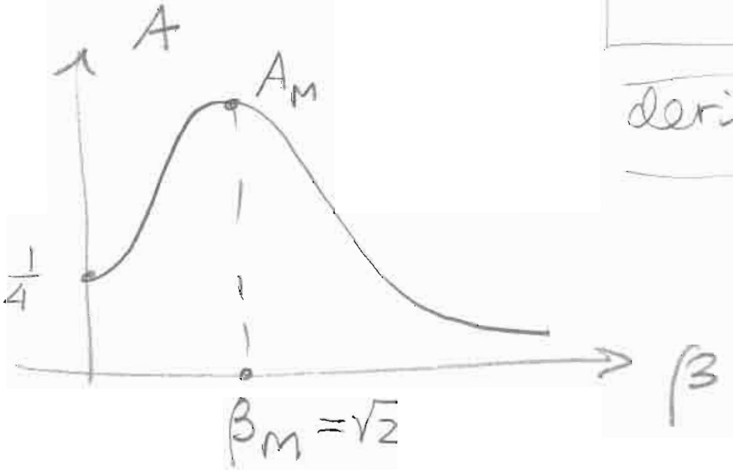
7

Responce ampl: $$A(\beta) = \frac{1}{\sqrt{(4-\beta^2)^2 + 4\beta^2}}$$

deriv $\frac{d}{d\beta^2} = 2(\beta^2 - 4) + 4 = 0$

$\beta = \sqrt{2}$

$$A_M = \frac{1}{\sqrt{12}}$$



$\frac{+}{4}$

4. Dynamical systems:

a) Take predator-prey model 1(d) with $a=1/4$. Write linearized systems for its 3 equilibria. Sketch local phase-plot near each equilibria, and the global phase portrait. **Extra**: Explain what happens to the system (equilibria et al) as parameter a changes from ¼ to 1.

$a=\frac{1}{4}$ Jacobian $A=\begin{bmatrix} 1-2ax-y & -x \\ y & (x-2) \end{bmatrix}$

Equil: (0, 0)	(4, 0)	(2, ½)
$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$	$\begin{bmatrix} -1 & -4 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} -1/2 & -2 \\ 1/2 & 0 \end{bmatrix}$
saddle	saddle	$p=\lambda^2+\frac{1}{2}\lambda+1 \Rightarrow \lambda_1=-\frac{1}{4}\pm i\sqrt{\frac{15}{16}}$ sp. sink

b) Determine which of the following fields is Hamiltonian and find its Hamiltonian function:
$\mathbf{F}=(x-2y,x+y^2)$; $\mathbf{G}=(y^2-x,y-2)$; $\mathbf{H}=(ax-y,x-2y)$

$\nabla\cdot \mathbb{F} = 1+2y \neq 0$ No

$\nabla\cdot \mathbb{G} = -1+1 = 0$ Yes

$\nabla\cdot \mathbb{H} = a-2 \overset{?}{=} 0$ Ham. iff $a=2$

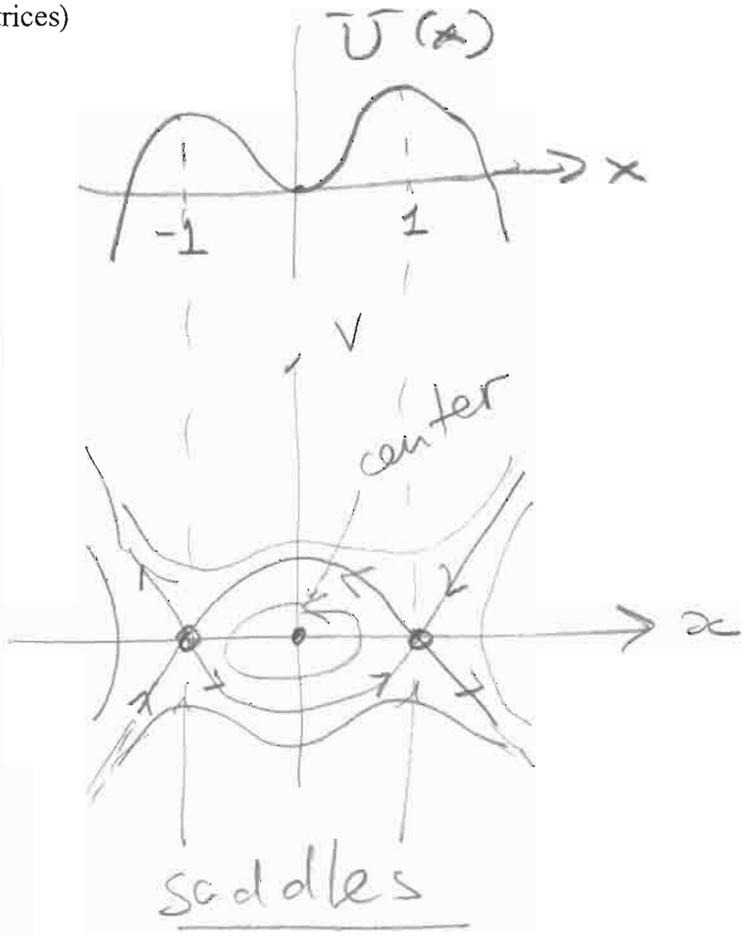
$\mathbb{G}: \begin{cases} 1^\circ\ h_y = y^2-x \Rightarrow & h=\frac{y^3}{2}-xy+C(x) \\ 2^\circ\ h_x=-y+2 \Rightarrow & -y+C'(x)=-y+2 \Rightarrow C(x)=2x \end{cases}$

$\Rightarrow \boxed{h(x,y)=\frac{y^3}{2}-xy+2x}$

Extra: for $\mathbb{H}$ (w. $a=2$) $h = 2xy - y^2/2 - x^2/2$

c) For mechanical system 1(c) sketch (i) potential function, (ii) phase-plane (nullclines and equilibria). Determine the equilibria types. **Extra:** write linearized system at equilibria and confirm your conclusions by the analysis of linear systems (Jacobian matrices)

8 $U(x) = \frac{x^2}{2} - \frac{x^4}{4}$



Extra

$F = \begin{pmatrix} v \\ x^3 - x \end{pmatrix}$

Jacobian: $A = \begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{bmatrix}$

+5

$(\pm 1, 0)$	$(0, 0)$
$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ saddle	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ center
$\lambda_{1,2} = \pm\sqrt{2}$	$\lambda_{1,2} = \pm i$