Section 7.5 Nonlinear systems, unlike linear systems, sometimes have periodic solutions, or **limit cycles**, that attract other nearby solutions.

- Several theorems specify conditions under which limit cycles do, or do not, exist.
- The **van der Pol equation** (written in system form)

$$x' = y, \qquad y' = -x + \mu\left(1 - x^2\right)y$$

is an important equation that illustrates the occurrence of a limit cycle.

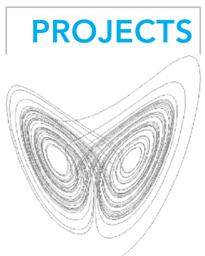
Section 7.6 In three or more dimensions there is the possibility that solutions may be **chaotic**. In addition to critical points and limit cycles, solutions may converge to sets of points known as **strange attractors**.

- The **Lorenz equations**, arising in a study of the atmosphere,

$$dx/dt = \sigma(-x + y), \qquad dy/dt = rx - y - xz, \qquad dz/dt = -bz + xy$$

provide an example of the occurence of chaos in a relatively simple three-dimensional nonlinear system.

PROJECTS



Project 1 Modeling of Epidemics

Infectious disease is disease caused by a biological agent (virus, bacterium, or parasite) that can be spread directly or indirectly from one organism to another. A sudden outbreak of infectious disease which spreads rapidly and affects a large number of people, animals, or plants in a particular area for a limited period of time is referred to as an *epidemic*. Mathematical models are used to help understand the dynamics of an epidemic, to design treatment and control strategies (such as a vaccination program or quarantine policy), and to help forecast whether an epidemic will occur. In this project, we consider two simple models which highlight some important principles of epidemics.

The SIR Model. Most mathematical models of disease assume that the population is subdivided into a set of distinct compartments, or classes. The class in which an individual resides at time t depends on that individual's experience with respect to the disease. The simplest of these models classifies individuals as either susceptible, infectious, or removed from the population following the infectious period (see Figure 7.P.1).

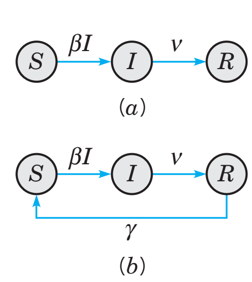


FIGURE 7.P.1 (*a*) The SIR epidemic model, and (*b*) the SIRS epidemic model.

Accordingly, we define the state variables

$$\begin{aligned}
S(t) &= \text{number of susceptible individuals at time } t,\\
I(t) &= \text{number of infected individuals at time } t,\\
R(t) &= \text{number of post-infective individuals removed from the}\\
&\quad\ \text{population at time } t \text{ (due to immunity, quarantine, or death).}
\end{aligned}$$

Susceptible individuals are able to catch the disease, after which they move into the infectious class. Infectious individuals spread the disease to susceptibles, and remain in the infectious class for a period of time (the infectious period) before moving into the removed class. Individuals in the removed class consist of those who can no longer acquire or spread the disease. The mathematical model (referred to as the SIR model) describing the temporal evolution of the sizes of the classes is based on the following assumptions:

1. The rate at which susceptibles become infected is proportional to the number of encounters between susceptible and infected individuals, which in turn is proportional to the product of the two populations, βSI. Larger values of β correspond to higher contact rates between infecteds and susceptibles.
2. The rate of transition from class I to class R is proportional to I, that is, νI. The biological meaning of ν is that $1/\nu$ is the average length of the infectious period.
3. During the time period over which the disease evolves there is no immigration, emigration, births, or deaths except possibly from the disease.

With these assumptions, the differential equations that describe the number of individuals in the three classes are

$$\begin{aligned}
S' &= -\beta I S,\\
I' &= \beta I S - \nu I, \qquad (1)\\
R' &= \nu I.
\end{aligned}$$

It is convenient to restrict analysis to the first two equations in Eq. (1) since they are independent of R,

$$\begin{aligned}
S' &= -\beta I S,\\
I' &= \beta I S - \gamma I. \qquad (2)
\end{aligned}$$

The SIRS Model. A slight variation in the SIR model results by assuming that individuals in the R class are temporarily immune, say for an average length of time $1/\gamma$, after which they rejoin the class of susceptibles. The governing equations in this scenario, referred to as the SIRS model, are

$$\begin{aligned}
S' &= -\beta I S + \gamma R,\\
I' &= \beta I S - \nu I, \qquad (3)\\
R' &= \nu I - \gamma R.
\end{aligned}$$

Project 1 PROBLEMS

1. Assume that $S(0) + I(0) + R(0) = N$, that is, the total size of the population at time $t = 0$ is N. Show that $S(t) + I(t) + R(t) = N$ for all $t > 0$ for both the SIR and SIRS models.
2. The triangular region $\Gamma = \{(S, I)\colon 0 \le S + I \le N\}$ in the SI-plane is depicted in Figure 7.P.2. Use an analysis based strictly on direction fields to show that no solution of the system (2) can leave the set Γ. More precisely, show that each point on the boundary of Γ is either a critical point of the system (2), or.else the direction field vectors point toward the interior of Γ or are parallel to the boundary of Γ.
3. If epidemics are identified with solution trajectories in which the number of infected individuals initially increases, reaches a maximum, and then decreases, use a nullcline analysis to show that an epidemic occurs if and only if $S(0) > \rho = \nu/\beta$. Assume that $\nu/\beta < 1$. Thus, $\rho = \nu/\beta$ is, in effect, a threshold value of susceptibles separating Γ into an epidemic region and a nonepidemic region. Explain how the size of the nonepidemic

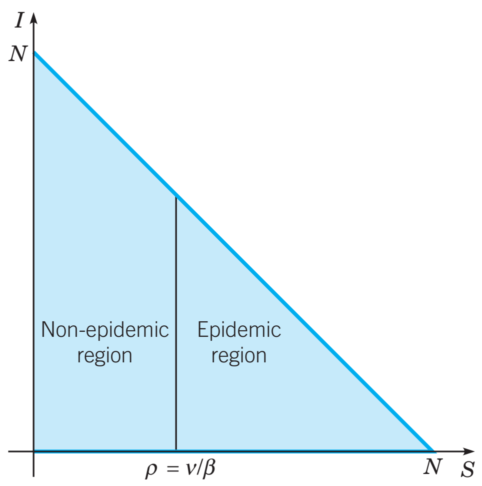


FIGURE 7.P.2 The state variables S and I for the SIR and SIRS models must lie in the region $\Gamma = \{(S, I) : 0 \le S + I \le N\}$.

region depends on contact rate and length of infection period.

ODEA **4.** Find an equation of the form $H(S, I) = c$ satisfied by the solutions of Eq. (2). Then construct a phase portrait within Γ for the system (2) consisting of trajectories emanating from points along the upper boundary $S + I = N$ of Γ corresponding to initial states in which $R(0) = 0$.

5. In the SIR system (1), describe qualitatively the asymptotic behavior of S, I, and R as $t \to \infty$. In particular, answer the question, "Does everyone get infected?" Then explain the statement, "The epidemic does not die out due to the lack of susceptibles, but rather due to a lack of infectives."

6. Vaccinated individuals are protected from acquiring the disease and are, in effect, removed from participating in the transmission of the disease. Explain how an epidemic can be avoided by vaccinating a sufficiently large fraction p of the population, but it is not necessary to vaccinate the entire population.

7. Use the equation $S + I + R = N$ to reduce the SIRS model (3) to a system of dimension 2. Then use the qualitative methods of Chapter 7 and numerical simulations to discover as much as you can about the properties of solutions of the system (3). Compare and contrast your findings with the properties of solutions of the SIR model. ODEA

Project 2 Harvesting in a Competitive Environment

Consider again the system (Eq. (2) of Section 7.3)

$$\begin{aligned} dx/dt &= x(\epsilon_1 - \sigma_1 x - \alpha_1 y), \\ dy/dt &= y(\epsilon_2 - \sigma_2 y - \alpha_2 x), \end{aligned} \tag{1}$$

which models competition between two species. To be specific, suppose that x and y are the populations of two species of fish in a pond, lake, or ocean. Suppose further that species x is a good source of nourishment, so that it is desirable to harvest members of x for food. Intuitively, it may seem reasonable to believe that if x is harvested too aggressively, then its numbers may be reduced to the point where it is no longer able to survive the competition with y and will decline to possible extinction. So the policy issue is how to determine a harvest rate that will provide useful food without threatening the long-term survival of the species. There are two simple models that have been used to investigate harvesting in a competitive situation, a constant-effort model and a constant-yield model. The first of these is described in Problems 1 through 3, and the second in Problem 4.

Project 2 PROBLEMS

1. Consider again the system

$$\begin{aligned} dx/dt &= x(1 - x - y), \\ dy/dt &= y(0.75 - y - 0.5x), \end{aligned} \tag{i}$$

which appeared in Example 1 of Section 7.3. A constant-effort model, applied to the species x alone, assumes that the rate of growth of x is altered by including the term $-Ex$, where E is a positive constant measuring the effort invested in harvesting members of species x. This assumption means that, for a given effort E, the rate of catch is proportional to the population x, and that for a given population x the rate of catch is proportional to the effort E. Based on this assumption, Eqs. (i) are replaced by

$$\begin{aligned} dx/dt &= x(1 - x - y) - Ex = x(1 - E - x - y), \\ dy/dt &= y(0.75 - y - 0.5x). \end{aligned} \tag{ii}$$

(a) For $E = 0$, the critical points of Eqs. (ii) are as in Example 1 of Section 7.3. As E increases, some critical points move while others remain fixed. Which ones move and how?

(b) For a certain value of E, denoted by E_0, the asymptotically stable node, originally at the point (0.5, 0.5), coincides with the saddle point (0, 0.75). Find the value of E_0.

ODEA **(c)** Draw a direction field and/or a phase portrait for $E = E_0$ and for values of E slightly less than and slightly greater than E_0.

(d) How does the nature of the critical point (0, 0.75) change as E passes through E_0?

(e) What happens to the species x for $E > E_0$?

2. Consider the system

$$\begin{aligned} dx/dt &= x(1 - x - y), \\ dy/dt &= y(0.8 - 0.6y - x), \end{aligned} \tag{iii}$$

which appeared in Example 2 of Section 7.3. If constant-effort harvesting is applied to species x, then the modified equations are

$$\begin{aligned} dx/dt &= x(1 - x - y) - Ex = x(1 - E - x - y), \\ dy/dt &= y(0.8 - 0.6y - x). \end{aligned} \tag{iv}$$

(a) For $E = 0$, the critical points of Eqs. (iv) are as in Example 2 of Section 7.3. As E increases, some critical points move while others remain fixed. Which ones move and how?

(b) For a certain value of E, denoted by E_0, the saddle point originally at (0.5, 0.5), coincides with the asymptotically stable node originally at (1, 0). Find the value of E_0.

ODEA **(c)** Draw a direction field and/or a phase portrait for $E = E_0$ and for values of E slightly less than, and slightly greater than, E_0. Estimate the basin of attraction for each asymptotically stable critical point.

(d) Consider the asymptotically stable node originally at (1, 0). How does the nature of this critical point change as E passes through E_0?

(e) What happens to the species x for $E > E_0$?

3. Consider the system (i) in Problem 1, and assume now that both x and y are harvested, with efforts E_1 and E_2, respectively. Then the modified equations are

$$\begin{aligned} dx/dt &= x(1 - E_1 - x - y), \\ dy/dt &= y(0.75 - E_2 - y - 0.5x). \end{aligned} \tag{v}$$

(a) When $E_1 = E_2 = 0$ there is an asymptotically stable node at (0.5, 0.5). Find conditions on E_1 and E_2 that permit the continued long-term survival of both species.

(b) Use the conditions found in part **(a)** to sketch the region in the E_1E_2-plane that corresponds to the long-term survival of both species. Also identify regions where one species survives but not the other, and a region where both decline to extinction.

4. A constant-yield model, applied to species x, assumes that dx/dt is reduced by a positive constant H, the yield rate. For the situation described by Eqs. (i), the modified equations are

$$\begin{aligned} dx/dt &= x(1 - x - y) - H, \\ dy/dt &= y(0.75 - y - 0.5x). \end{aligned} \tag{vi}$$

(a) For $H = 0$, the x-nullclines are the lines $x = 0$ and $x + y = 1$. For $H > 0$ show that the x-nullcline is a hyperbola whose asymptotes are $x = 0$ and $x + y = 1$.

(b) How do the critical points move as H increases from zero?

(c) For a certain value of H, denoted by H_c, the asymptotically stable node originally at (0.5, 0.5) coincides with the saddle point originally at (0, 0.75). Determine the value of H_c. Also determine the values of x and y where the two critical points coincide.

(d) Where are the critical points for $H > H_c$? Classify them as to type.

(e) What happens to species x for $H > H_c$? What happens to species y?

(f) Draw a direction field and/or phase portrait for $H = H_c$ and for values of H slightly less than, and slightly greater than, H_c. ODEA

EQA

P1: PBU/OVY P2: PBU/OVY QC: PBU/OVY T1: PBU
GTBL021-07 GTBL021-Branan-v12 October 18, 2006 20:53

Project 3 The Rössler System

The system

$$x' = -y - z, \quad y' = x + ay, \quad z' = b + z(x - c), \tag{1}$$

where a, b, and c are positive parameters, is known as the Rössler[6] system. It is a relatively simple system, consisting of two linear equations and a third equation with a single quadratic nonlinearity. In the following problems, we ask you to carry out some numerical investigations of this system, with the goal of exploring its period-doubling property. To simplify matters set $a = 0.25$, $b = 0.5$, and let $c > 0$ remain arbitrary.

Project 3 PROBLEMS

ODEA **1.** **(a)** Show that there are no critical points when $c < \sqrt{0.5}$, one critical point for $c = \sqrt{0.5}$, and two critical points when $c > \sqrt{0.5}$.
(b) Find the critical point(s) and determine the eigenvalues of the associated Jacobian matrix when $c = \sqrt{0.5}$ and when $c = 1$.
(c) How do you think trajectories of the system will behave for $c = 1$? Plot the trajectory starting at the origin. Does it behave the way that you expected?
(d) Choose one or two other initial points and plot the corresponding trajectories. Do these plots agree with your expectations?

ODEA **2.** **(a)** Let $c = 1.3$. Find the critical points and the corresponding eigenvalues. What conclusions, if any, can you draw from this information?
(b) Plot the trajectory starting at the origin. What is the limiting behavior of this trajectory? To see the limiting behavior clearly, you may wish to choose a t-interval for your plot so that the initial transients are eliminated.
(c) Choose one or two other initial points and plot the corresponding trajectories. Are the limiting behavior(s) the same as in part **(b)**?
(d) Observe that there is a limit cycle whose basin of attraction is fairly large (although not all of xyz-space). Draw a plot of x, y, or z versus t and estimate the period of motion around the limit cycle.

3. The limit cycle found in Problem 2 comes into existence as a result of a Hopf bifurcation at a value c_1 of c between 1 and 1.3. Determine, or at least estimate more precisely, the value of c_1. There are several ways in which you might do this.
(a) Draw plots of trajectories for different values of c. ODEA
(b) Calculate eigenvalues at critical points for different values of c.
(c) Use the result of Problem 3**(b)** in Section 7.6.

4. **(a)** Let $c = 3$. Find the critical points and the corresponding eigenvalues. ODEA
(b) Plot the trajectory starting at the point $(1, 0, -2)$. Observe that the limit cycle now consists of two loops before it closes; it is often called a 2-cycle.
(c) Plot x, y, or z versus t and show that the period of motion on the 2-cycle is very nearly double the period of the simple limit cycle in Problem 2. There has been a period-doubling bifurcation of cycles for a certain value of c between 1.3 and 3.

5. **(a)** Let $c = 3.8$. Find the critical points and the corresponding eigenvalues. ODEA
(b) Plot the trajectory starting at the point $(1, 0, -2)$. Observe that the limit cycle is now a 4-cycle. Find the period of motion. Another period-doubling bifurcation has occurred for c between 3 and 3.8.
(c) For $c = 3.85$ show that the limit cycle is an 8-cycle. Verify that its period is very close to eight times the period of the simple limit cycle in Problem 2.
Note: As c increases further, there is an accelerating cascade of period-doubling bifurcations. The bifurcation values of c converge to a limit, which marks the onset of chaos.

[6] See the book by Strogatz for a more extensive discussion and further references.