M224 homework solutions

February 3, 2012

3^{rd} edition

Exercise 1.1 problem 4 (5)

(a) At equilibrium, $\frac{dP}{dt} = 0$.

$$(1 - \frac{P}{200})(\frac{P}{500})P = 0,$$

which has solution $P = 0, \quad P = 50, \quad P = 200$.

(b) Increasing population means $\frac{dP}{dt} > 0$

$$(1 - \frac{P}{200})(\frac{P}{500})P > 0,$$

Since population is counting number, it should be non-negative,$P > 0$. Therefore

$$(1 - \frac{P}{200})(\frac{P}{500}) > 0,$$

Which is a quadratic inequality with solution $50 < P < 200$.

(c) Here decreasing population will mean $\frac{dP}{dt} < 0$.

$$(1 - \frac{P}{200})(\frac{P}{500}) < 0,$$

This happens at $0 < P < 50$ or $P > 200$.

Exercise 1.1 problem 6 (8)

(a) $\frac{dA}{dt} = kA$, solve by separation of variable to get $A(t) = A_0 e^{kt}$, t being the time in years from 1939.

(b) Taking 1939 as t=0, we have $A_0 = 32800$.
Now by 1944, $t = 1944 - 1939 = 5$years. So

$$A(5) = 32800e^{5k},$$
$$55800 = 32800e^{5k},$$
$$\ln(\frac{558}{328}) = 5k,$$
$$k = \frac{1}{5}\ln(\frac{558}{328}) = 0.1063.$$

(c) Now by 2010, $t = 2010 - 1939 = 71$years,

$$A(71) = 32800e^{71(0.1063)} = 6.2176 \times 10^7,$$

by 2050, $t = 2050 - 1939 = 111$years,

$$A(111) = 32800e^{111(0.1063)} = 4.3676 \times 10^9,$$

by 2100, $t = 2100 - 1939 = 161$years,

$$A(161) = 32800e^{161(0.1063)} = 8.8822 \times 10^{11},$$

(d) The prediction does not seem to predict the data correctly so the model should not be trusted.
On the other hand if we used 1949 value to compute the k, then the model better approximates the data.

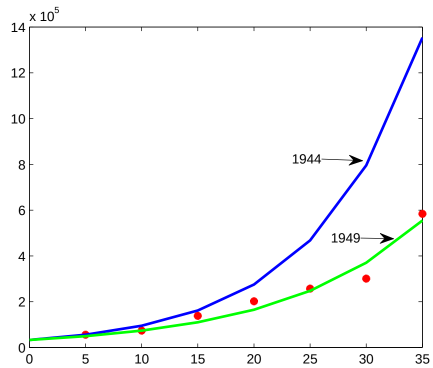

Figure 1: Graph of data(red) and model using 1949 to compute k(green) and 1944 to compute k (blue)

Exercise 1.1 problem 12 (6)

(a) $t_{\frac{1}{2}} = 5230$ model is

$$A(t) = A_0 e^{kt}. \tag{1}$$

At $t_{\frac{1}{2}} = 5230$, $A(t_{\frac{1}{2}}) = \frac{A_0}{2}$. So

$$\begin{aligned} A(t_{\frac{1}{2}}) &= A_0 e^{kt_{\frac{1}{2}}}, \\ \frac{\cancel{A_0}}{2} &= \cancel{A_0} e^{5230k}, \\ \frac{1}{2} &= e^{5230k}, \\ \ln(\frac{1}{2}) &= 5230k, \\ k &= -\frac{\ln(2)}{5230} \end{aligned} \tag{2}$$

Note: some will prefer to introduce the negative sign from the definition of the model, in that case k will turn out to be positive.

When 88% is left, we will have

$$\frac{88}{100}\cancel{A_0} = \cancel{A_0}e^{kt},$$
$$\frac{88}{100} = e^{kt},$$
$$\ln(\frac{88}{100}) = kt,$$
$$t = \frac{\ln(\frac{88}{100})}{k} = 964.54\text{years}.$$

(b) Likewise

$$t = \frac{\ln(\frac{12}{100})}{k} = 15998\text{years}.$$

(c) Likewise

$$t = \frac{\ln(\frac{2}{100})}{k} = 29517\text{years}.$$

(d) Likewise

$$t = -\frac{\ln(\frac{12}{100})}{k} = 152.44\text{years}.$$

Exercise 1.1 problem 18 (6)

(a) We will try using the logistic model since the population seems to level off after 1956 at 64. So we will write

$$\frac{dP}{dt} = kP(1 - \frac{P}{N}),$$

where $P(t)$ is the population after t years and k is population growth rate, N the carrying capacity.

(b) Here the parameters are N and k. since we said that the population levels off at 64, so $N = 64$
To find k, we can choose any two consecutive years and use the data set to compute the slope. This slope should be the same as the $\frac{dP}{dt}$ evaluated at that year. So choosing 1947 as t=0, we

$$\frac{40-34}{1-0} \approx k \cdot 34(1 - \frac{34}{64}),$$
$$6 = 34 \cdot (\frac{30}{64})k,$$
$$k \approx 0.38.$$

(c) The population reaches equilibrium since 1956 at 64. Therefore today's predicted population will be 64.

Exercise 1.2 problem 10 (5)

$$
\begin{aligned}
\frac{dx}{dt} &= (1+x^2), \\
dx &= (1+x^2)tdt, \qquad \text{here we multiplied through by} \quad dt \\
\frac{dx}{(1+x^2)} &= dt, \qquad \text{here we divided through by} \quad (y^2+1) \\
\int \frac{dx}{(1+x^2)} &= \int tdt, \\
\tan^{-1}(x) &= t+k, \\
x &= \tan(t+k).
\end{aligned}
$$

So the solution to the differential equation is

$$x = \tan(t+k).$$

Exercise 1.2 problem 25 (5)

$$
\begin{aligned}
\frac{dy}{dt} &= -y^2, \\
dy &= y^2 dt, \qquad \text{here we multiplied through by} \quad dt \\
-\frac{dy}{(y^2)} &= dt, \qquad \text{here we divided through by} \quad (-y^2) \\
\int -\frac{dy}{y^2} &= \int dt, \\
\frac{1}{y} &= t+k, \\
y &= \frac{1}{t+k}.
\end{aligned}
$$

Now using the initial condition $y(0) = \frac{1}{2}$,

$$
\begin{aligned}
y(0) = \frac{1}{k} &= \frac{1}{2}, \\
k &= 2,
\end{aligned}
$$

So the solution to the initial value problem is

$$y = \frac{1}{t+2}.$$

Exercise 1.2 problem 31 (5)

Solve $\frac{dy}{dt} = (y^2+1)t, \qquad y(0) = 1.$
We will solve this by separation of variables

$$\begin{aligned}
\frac{dy}{dt} &= (y^2+1)t, \\
dy &= (y^2+1)tdt, \qquad \text{here we multiplied through by} \quad dt \\
\frac{dy}{(y^2+1)} &= tdt, \qquad \text{here we divided through by} \quad (y^2+1) \\
\int \frac{dy}{(y^2+1)} &= \int tdt, \\
\tan^{-1}(y) &= \frac{1}{2}t^2 + k, \\
y &= \tan(\frac{1}{2}t^2 + k).
\end{aligned}$$

Now using the initial condition $y(0) = 1$,

$$\begin{aligned}
y(0) &= \tan(\frac{1}{2}(0)^2 + k), \\
1 &= \tan(k), \\
k &= \tan^{-1}(1) = \frac{\pi}{4},
\end{aligned}$$

So the solution to the initial value problem is

$$y = \tan(\frac{1}{2}t^2 + \frac{\pi}{4}).$$

4^{th} edition

Exercise 1.2 problem 43 (6)

(a)

$$\begin{aligned}
m\frac{dv}{dt} &= mg - kv^2, \\
m\frac{dv}{dt} &= k(\frac{mg}{k} - v^2), \\
\frac{dv}{(\frac{mg}{k} - v^2)} &= \frac{k}{m}dt, \\
\int \frac{dv}{(\frac{mg}{k} - v^2)} &= \int \frac{k}{m}dt,
\end{aligned}$$

$$\frac{1}{2}\sqrt{\frac{k}{mg}} \ln|\frac{v + \sqrt{\frac{mg}{k}}}{v - \sqrt{\frac{mg}{k}}}| = \frac{k}{m} + C. \tag{3}$$

Taking ln on both sides and solving for v gives

$$v = \sqrt{\frac{mg}{k}} \left(\frac{Ce^{2\sqrt{\frac{kg}{m}}t} - 1}{Ce^{2\sqrt{\frac{kg}{m}}t} + 1} \right)$$

(b)

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \{ \sqrt{\frac{mg}{k}} \left(\frac{Ce^{2\sqrt{\frac{kg}{m}}t} - 1}{Ce^{2\sqrt{\frac{kg}{m}}t} + 1} \right) \} = \sqrt{\frac{mg}{k}}.$$