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Repeated eigenvalues: $\boxed{\lambda_1 = \lambda_2}$

$1^\circ \quad A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI$ - scalar

2 cases:

$2^\circ \quad A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$

Case 1°: Any X is eigenvector: $A \cdot X = aX$

$p = (\lambda - a)^2 \Rightarrow \boxed{\lambda_1 = \lambda_2 = a}$

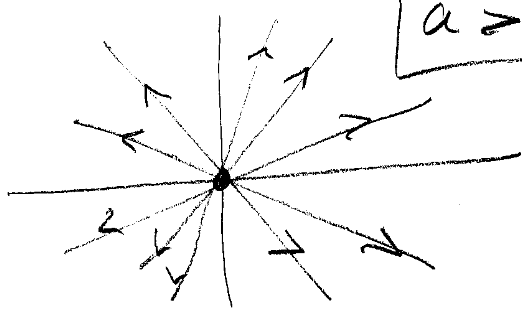
a	a
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\leftarrow$ indep. E-vectors

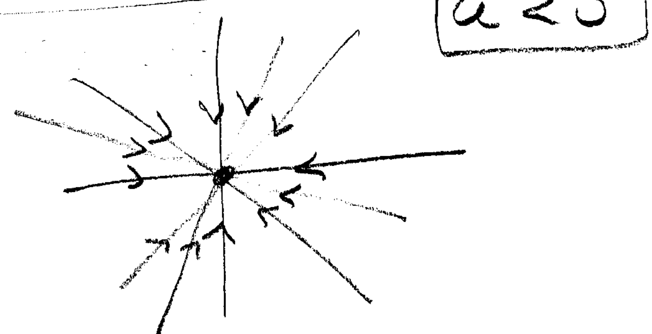
Fund. pair: $\{ \underbrace{e^{at}\begin{pmatrix} 1 \\ 0 \end{pmatrix}} ; \underbrace{e^{at}\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \}$

$GS = c_1 \, Y_1(t) + c_2 \, Y_2(t)$

$\boxed{a > 0}$



$\boxed{a < 0}$



Case 2°: $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \Rightarrow$

a	a
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	?

$Y_1 = e^{at}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, Y_2 - ??

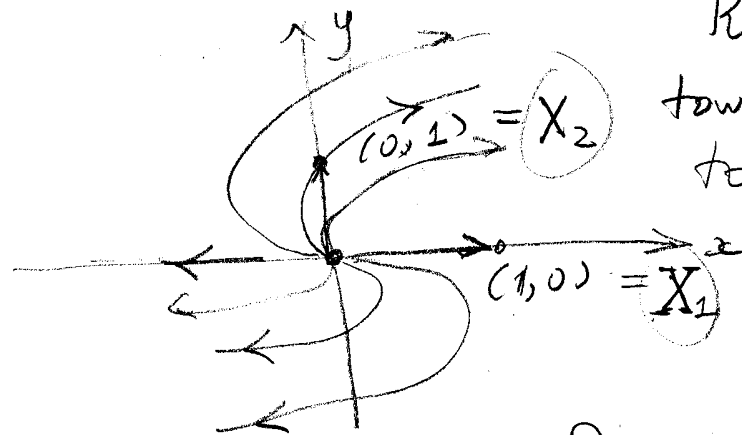
Multiplier

$$DS \begin{cases} \dot{x} = ax + y \\ \dot{y} = ay \end{cases} \Rightarrow \begin{cases} x(t) = c_1 e^{at} + c_2 \boxed{\underbrace{\int_0^t e^{a(t-s)} e^{as}\, ds}_{te^{at}}} \\ y(t) = c_2 e^{at} \end{cases}$$

$$\Rightarrow GS \quad \begin{pmatrix} x \\ y \end{pmatrix} = e^{at}\begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix} = c_1 e^{at}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{at}\left[t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

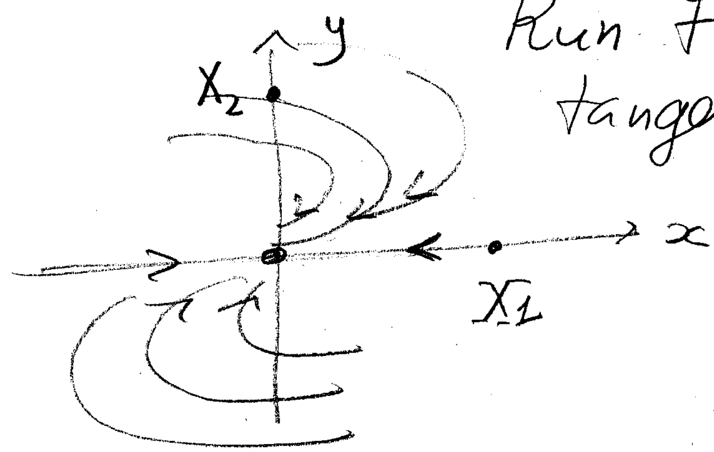
IVP: $$\boxed{\begin{pmatrix}x\\y\end{pmatrix} = e^{at}\begin{pmatrix}x_0 + y_0 t\\ y_0\end{pmatrix}}$$

$\boxed{a>0}$



Run from $\vec{O}$ towards X_2 tangent to "$-X_1$"

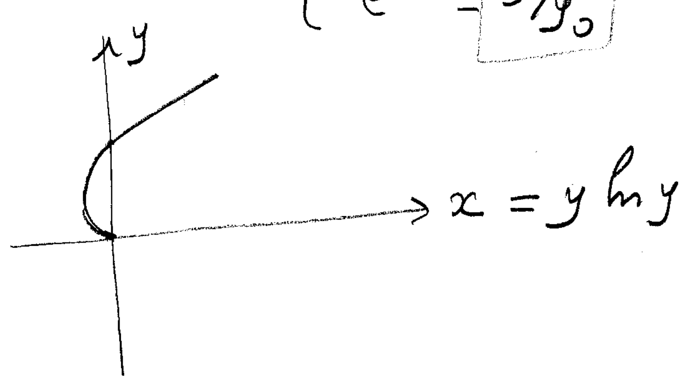
$\boxed{a<0}$



Run from X_2 to $\vec{O}$ tangent to $+X_1$

From IVP:

$$\begin{cases} x = y\left(\dfrac{x_0}{y_0} + \dfrac{1}{a}\ln y/y_0\right) = \boxed{By + \frac{1}{a} y \ln y} \\ e^{at} = \boxed{y/y_0} \end{cases}$$



For a general matrix A with repeated eigen λ, try solution

(1) $\boxed{\underline{Y}(t) = e^{\lambda t} X(t)}$ w. polynomial $X(t) = X_0 + X_1 t + \ldots$

– an "extended form" of line solut. $\boxed{e^{\lambda t} X_0}$

Need λ, $X(t)$ – ?

1) $\Rightarrow$ $\underline{Y}'(t) = \cancel{e^{\lambda t}}(X'(t) + \lambda X(t)) = \cancel{e^{\lambda t}} A \cdot X(t)$

Get new DE for $X(t)$:

(2) $\boxed{X'(t) = (A - \lambda I) \cdot X(t)}$

2) Try $X(t) = X_0 + X_1 t + \ldots$ (for 2×2 A linear polynom. suffice)

Collect powers $t^0, t, \ldots$

$$\begin{array}{l|l}
t & (A-\lambda I)\cdot X_1 = 0 \Rightarrow \\
t^0 = 1 & (A - \lambda I)\cdot X_0 = X_1 \Rightarrow
\end{array}
\quad
\boxed{\begin{array}{l} X_1 \text{ – eigenvector} \\ X_0 \text{ – assoc. v. to } X_1 \end{array}}$$

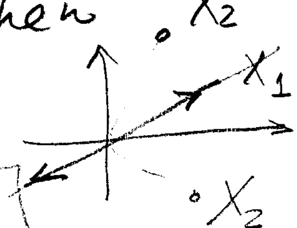
Associate vector to eigenvector

Eigenvector: $\boxed{(A-\lambda_1 I)\cdot X_1 = \vec{0}}$ (1)

Assoc. vector: $\boxed{(A-\lambda_1 I)\cdot X_2 = X_1}$ (2)

...

In 2D: if A has double eigen λ_1 (non-scalar!) & single X_1; then any $X_2 \neq c\,X_1$ is associate;



Procedure: 1° Pick $\boxed{X_2 \neq c\,X_1}$

2° Compute $\boxed{X_1 = (A-\lambda_1 I)\cdot X_2}$

$$3^\circ \quad Y_2(t) = e^{\lambda_1 t}\left(X_2 + t X_1\right)$$

Example: $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \Rightarrow p(\lambda) = (\lambda+1)^2$

$\lambda_1 = -1$	?
$X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$	?

$\boxed{Y_1(t) = e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix};}$

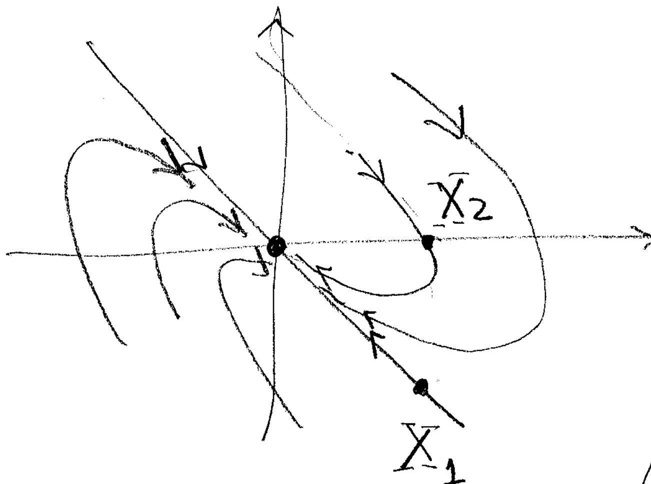
1° Take $X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2° Compute $X_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

3° $\boxed{Y_2(t) = e^{-t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]}$

Phase-plane :

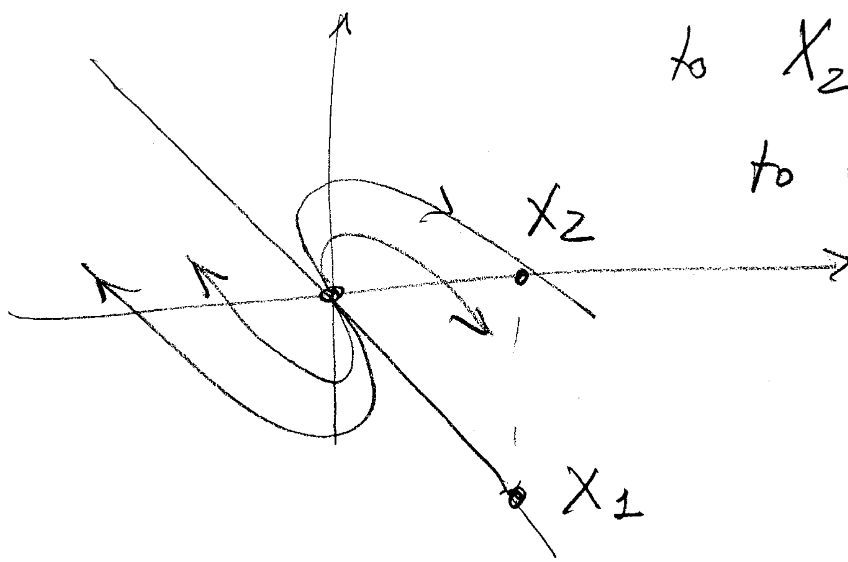
$\boxed{a<0}$



Run from X_2 to $\vec{O}$ tangent to $+X_1$

$\boxed{a>0}$

Run from $\vec{O}$ to X_2 tangent to $-X_1$



Zero eigenvalue

3. Migration problem: $\underset{x}{\text{(A)}} \underset{b}{\overset{a}{\rightleftarrows}} \underset{y}{\text{(B)}}$ – populations

$$\begin{cases} \dot{x} = -ax + by; \\ \dot{y} = ax - by; \end{cases} \quad A = \begin{bmatrix} -a & b \\ a & -b \end{bmatrix}; \quad p = \lambda^2 + (a+b)\lambda$$

0	$-(a+b)$
$\binom{b}{a}$	$\binom{-1}{1}$

GS: $\underline{x}(t) = c_1 \binom{a}{b} + c_2 \binom{-1}{1} e^{-(a+b)t}$

IVP: for $\underline{x}_0 = \binom{x_0}{P - x_0} \Rightarrow \begin{cases} x = P\frac{b}{a+b} + \left(x_0 - \frac{bP}{a+b}\right) e^{-(a+b)t} \\ y = P\frac{a}{a+b} + \left(y_0 - \frac{aP}{a+b}\right) e^{-(a+b)t} \end{cases}$

total popul.

Limit: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \underset{t \to \infty}{\longrightarrow} P \begin{pmatrix} b/a+b \\ a/a+b \end{pmatrix}$

population distributed according to the migration rates $x = b$-fraction, $y = a$-fraction