M224

D. Gurarie

Other techniques

1. Change of variable.

$$\boxed{y' = f(y,t)} \xrightarrow{y \to z = y - at} \boxed{z' = f(z+at,\, t) - a}$$

Ex: (i) $\boxed{y' = (y+at)^2 - a} \xrightarrow{z = y+at} \boxed{z' = z^2}$

$$y = \frac{y_0}{1 - y_0 t} - at \quad \leftarrow \quad z = \frac{z_0}{1 - z_0 t} \ : \ GS$$

(ii) $\boxed{y' = \dfrac{ay + bx}{cy + dx}}$; $y = y(x)$ - ?

$$\frac{ay+bx}{cy+dx} = \frac{a\,y/x + b}{c\,y/x + d} ; \quad \text{Change: } y \to u = y/x$$

$$\Rightarrow \quad (ux)' = \frac{au+b}{cu+d} \Rightarrow \boxed{x u' = \underbrace{\frac{au+b}{cu+d} - u}_{f(u)}} \text{ separ.}$$

Problem: Find general solution & plot slope field & solution curves for $y' = \dfrac{y+3x}{y-x}$ (use partial fractions & Mathematica)

2. Linear approximation (linearization)

$\boxed{y' = f(y)}$ or $\boxed{y' = f(y) + b(t)}$

Equilibrium: $f(y_0) = 0$, change $y \to u = y - y_0$

$\boxed{u' = \underbrace{f'(y_0)}_{m} u}$ or $\boxed{u' = \underbrace{f'(y_0)}_{m} u + b(t)}$

Ex: $f(y) = y(1-y)$

$\nearrow \boxed{y=0}$ $u' = u + \ldots$

$\searrow \boxed{y=1}$ $u' = -u + \ldots$

} linearized eq-ns

Qualitative analysis (sandwitch)

$$\text{IVP}\begin{cases} y' + 2y = b(t); & \boxed{-1<b<1} \\ y(0) = y_0; \end{cases} \rightarrow \text{complicated solut. } y(t) = .?$$

Ask for estimates $\ldots \le y(t) \le \ldots$?

Use approximate DEs

$$\begin{cases} u' + 2u = 1; \; u(0) = y_0 \longrightarrow u(t) = \frac{1}{2} + \left(y_0 - \frac{1}{2}\right)e^{-2t} \\ v' + 2u = -1; \; v(0) = y_0 \longrightarrow v(t) = -\frac{1}{2} + \left(y_0 + \frac{1}{2}\right)e^{-2t} \end{cases}$$

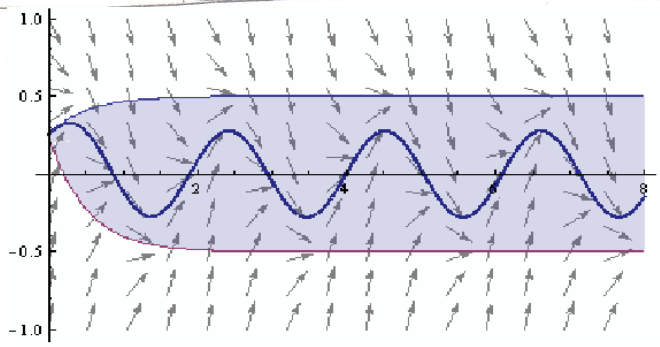
$$\boxed{v(t) \le y(t) \le u(t)}$$

General principle:

$$y' = \begin{cases} -2y + 1 = f_1(y,t) \\ -2y + b(t) = \boxed{f(y,t)} \\ -2y - 1 = f_2(y,t) \end{cases}$$

If $f_2(y,t) \le f(y,t) \le f_1(y,t)$ for all (y,t), then any solution

$$\boxed{y_2(t) \le y(t) \le y_1(t)}$$



Same idea applies top other (NL) DE, e.g.

(2) y'=y(1-y) +b*cos(w*t)

Problem: Find sandwitch for (2) with b=.2, y0 = .5. Plot it using DEPlot

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$$\boxed{y' = y^2 - t}$$

Change: $y = \pm\sqrt{t} + u \implies$

$$u' \pm \frac{1}{2\sqrt{t}} = (u \pm \sqrt{t})^2 - t = \cancel{u^2} \pm 2\sqrt{t}\, u$$

$u \ll \sqrt{t}$

$\implies$ LDE $\boxed{u' \mp 2\sqrt{t}\, u = \mp \frac{1}{2\sqrt{t}}}$ Linearized equations about parabolae +-sqrt(t)

$(+)$ $u = e^{\frac{4}{3} t^{3/2}} \left[c + \int_0^t \frac{e^{-\frac{4}{3} s^{3/2}}}{2\sqrt{s}} \right] = c e^{\frac{4}{3} t^{3/2}} + \frac{1}{2\sqrt{t}} \{1 + \ldots\}$

$(-)$ $u = e^{-\frac{4}{3} t^{3/2}} \left[c + \int_0^t \frac{e^{\frac{4}{3} s^{3/2}}}{2\sqrt{s}} \right] = c e^{-\frac{4}{3} t^{3/2}} +$

$$+ \frac{1}{2\sqrt{t}} \left\{ 1 + \frac{a_1}{t^{3/2}} + \frac{a_2}{t^3} + \ldots \right.$$

$\implies$ stable $\boxed{y = \sqrt{t} + \frac{1}{2\sqrt{t}} + \ldots + c e^{-\frac{4}{3} t^{3/2}}}$