Math. 224 supplement. Forced oscillators

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Few examples of damped and undamped forced oscillators

1 Undamped oscillators

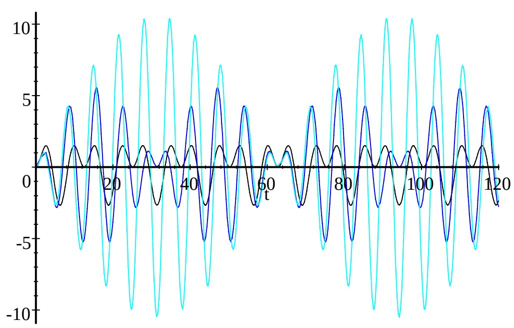
Differential equation: $mx'' + kx = F_0 \cos \omega t$. General solution

$$x(t) = \frac{F_0}{k - \omega^2 m} \cos \omega t + C_1 \cos \beta t + C_2 \sin \beta t$$

Here Ω is the forcing frequency, $\beta = \sqrt{\frac{k}{m}}$ - the proper frequency of oscillations. The IVP $\begin{matrix} x(0) = 0 \\ x'(0) = 0 \end{matrix}$ has solution

$$x(t) = \frac{F_0}{-k + \omega^2 m} (\cos \beta t - \cos \omega t) = 2A \sin \left(\frac{\beta + \omega}{2} \right) t \sin \left(\frac{\beta - \omega}{2} \right) t$$

with amplitude factor $A = \frac{F_0}{-k + \omega^2 m}$. Plot solutions for $\omega = .5\beta, .8\beta, .9\beta$



To show the modulation we write solution as

$$x(t) = 2\frac{F_0}{m}\frac{\sin\left(\frac{\beta-\omega}{2}\right)t}{(\beta-\omega)}\frac{\sin\left(\frac{\beta+\omega}{2}\right)t}{\beta+\omega}$$

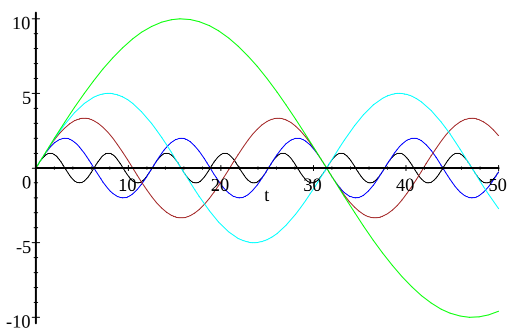
Factor $2\frac{F_0}{m}\frac{\sin\left(\frac{\beta-\omega}{2}\right)t}{(\beta-\omega)}$ represents slowly oscillating *modulation* part. Its frequency $\nu = \frac{\beta-\omega}{2}$ goes to 0 as $\omega \to \beta$, hence period increases

$$T = \frac{2\pi}{\nu} = \frac{4\pi}{\beta-\omega} \to \infty$$

In the above example $T = \frac{8\pi}{\beta}; \frac{20\pi}{\beta}; \frac{40\pi}{\beta}$. Amplitude of modulations is also increasing

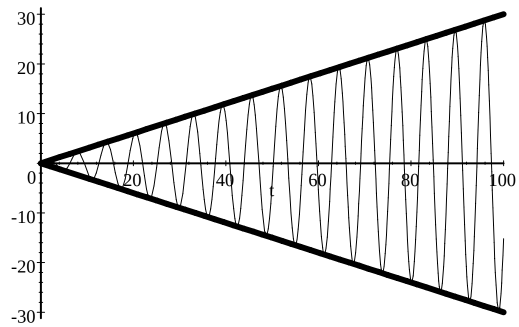
$$A = \max_t \frac{\sin \nu t}{\nu} = \frac{1}{\nu}$$

Here modulation function $\frac{\sin \nu t}{\nu}$ is plotted for $\nu = 1, .5, .3, .2, .1$



In the limit $\omega = \beta$ we get resonant solution

$$x(t) = \lim_{\Omega\to\omega} 2\frac{F_0}{m}\frac{\sin\left(\frac{\beta-\omega}{2}\right)t}{(\beta-\omega)}\frac{\sin\left(\frac{\beta+\omega}{2}\right)t}{\beta+\omega} = \left(\frac{F_0}{m\beta}\right)t\sin\beta t$$



so its amplitude increases linearly in time.

2 Damped oscillators

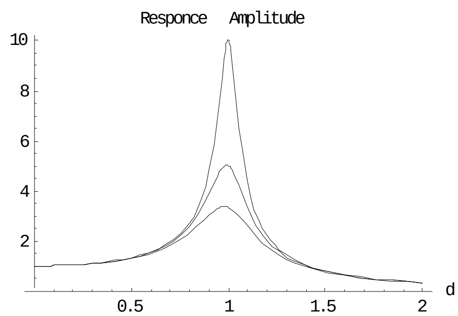
Differential equation: $mx'' + dx' + kx = F_0 \cos \omega t$. Particular (response) solution

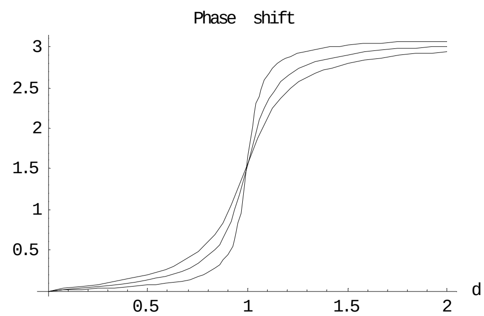
$$
\begin{aligned}
x_p(t) &= F_0 \operatorname{Re}\left(\frac{e^{i\omega t}}{(k-\omega^2 m) + id\omega}\right) = F_0 \frac{(k-\omega^2 m)\cos \omega t + d\omega \sin \omega t}{(k-\omega^2 m)^2 + d^2\omega^2} \\
&= A \cos \omega (t - \phi)
\end{aligned}
$$

whose amplitude and phase-shift

$$
\begin{aligned}
A &= \frac{1}{\sqrt{(k-\omega^2 m)^2 + d^2\omega^2}} \\
\phi &= \frac{1}{\omega} \tan^{-1}\left(\frac{d\omega}{(k-\omega^2 m)}\right)
\end{aligned}
$$

plotted below for several values of $d = .1; .3; .5$ with $k = m = 1$





Maximal responce amplitude (for $A_{\max}$) is found to be to

$$\omega_{\max} = \sqrt{(k^2 - d^2/2)/m}$$

Free damped oscillations depend on the characteristic roots

$$m\lambda^2 + d\lambda + k = 0 \Rightarrow \lambda = -\frac{d}{2m} \pm \sqrt{\frac{d^2}{4m^2} - \frac{k}{m}} \tag{1}$$

2.1 Underdamped case

If friction is small: $\frac{d^2}{4m^2} < \frac{k}{m}$ the roots are complex

$$\begin{aligned} \lambda &= -\alpha \pm i\beta \\ \alpha &= \frac{d}{2m} \text{ -exponential damping} \\ \beta &= \sqrt{\frac{k}{m} - \frac{d^2}{4m^2}} \text{ - frequency} \end{aligned}$$

$$x(t) = x_p(t) + e^{-\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$$

Example: Take oscillator model: $3x'' + dx' + 4x = 0$ with friction coefficient d . Exact solution is :

$$x(t) = C_1 \exp\left(-\frac{1}{6}\left(d - \sqrt{d^2 - 48}\right)t\right) + C_2 \exp\left(-\frac{1}{6}\left(d + \sqrt{(d^2 - 48)}\right)t\right)$$

Take small $d = .5$

$$x(t) = e^{-.08\,3t}(C_1 \cos 1.1517t + C_2 \sin 1.1517t)$$

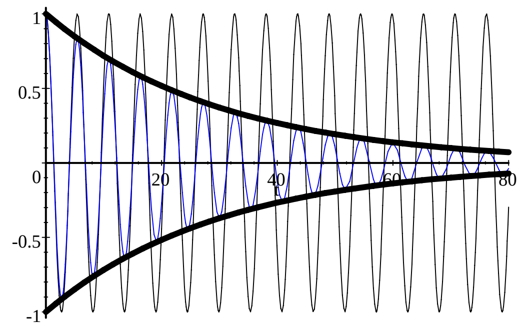
So the exponential decay rate $\alpha = \frac{d}{2m} = .08\,3$ and frequency

$$\beta = \frac{1}{6}\sqrt{48 - d^2} = 1.\,1517 < \omega = \sqrt{\frac{4}{3}} = 1.\,1547$$

decreases compared to free (undamped) oscillations.

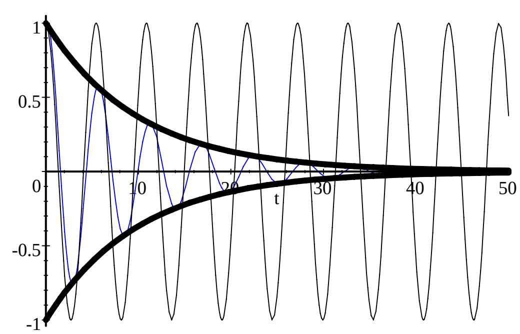
We look at the effect of damping on solution of IVP: $\begin{array}{c} x(0) = 1 \\ x'(0) = 0 \end{array}$

$$\begin{array}{ll} x(t) = \cos\sqrt{\frac{4}{3}}t & \text{undamped} \\ x(t) = e^{-.08\,3t}\left(\cos 1.\,1517t + .029 \sin 1.\,1517t\right) & \text{damped} \end{array}$$



It shows exponential fall-off of the amplitude but no significant change in frequency of oscillations. To see the latter increase the damping coefficient $\varepsilon = 1.5$. Now the frequency differs appreciably $\beta = 1.\,1273 < \omega = 1.\,1547$. Same initial data $\begin{array}{c} x(0) = 1 \\ x'(0) = 0 \end{array}$ gives

$$x(t) = e^{-.\,25t}\left(\cos 1.\,1273t + .\,22177 \sin 1.\,1273t\right)$$



2.2 Overdamped case

If friction is large enough: $\frac{d^2}{4m^2} \geq \frac{k}{m}$ or $d^2 \geq 4km$ - two roots become real-negative

$$\lambda_1 = -\frac{d}{2m} - \sqrt{\frac{d^2}{4m^2} - \frac{k}{m}} < \lambda_2 = -\frac{d}{2m} + \sqrt{\frac{d^2}{4m^2} - \frac{k}{m}} < 0$$

and unforced solution is a combination of two negative exponentials

$$x(t) = C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t}$$

In the above example $d \geq 2\sqrt{3 \cdot 4} = 6.9282$ -critical value. So friction $\varepsilon = 7$ gives

$$\lambda_1 = -\frac{4}{3}; \lambda_2 = -1$$

IVP-solution for $\begin{array}{c} x(0) = -1 \\ x'(0) = 3 \end{array}$ is $x(t) = -6e^{-\frac{4}{3}t} + 5e^{-t}$. No oscillations!