Bifurcation problem

D. GuRARIE

Family of DE/DS $\boxed{Y' = F(Y, b, \ldots)}$

Parameters $\{b, \ldots\}$

Q: 1) What happens to phase-space as b changes?

2) What —"— to equilibria ... ?

Ans: I) Equil. appear/disappear

II) Change type (stable/unstable; ...)

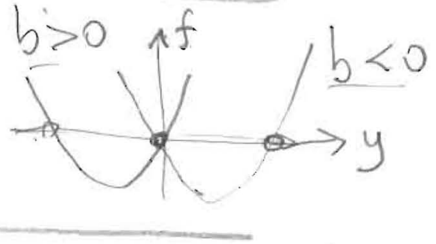
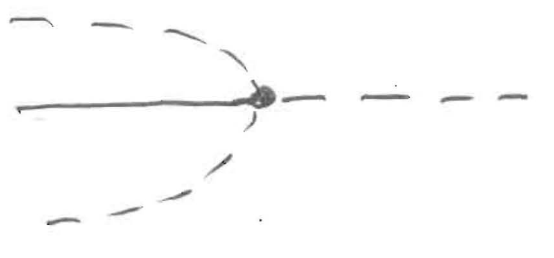
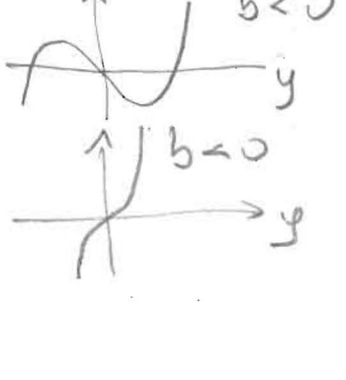
III) Other: "stable equil." ⟶ "limit cycle"

...

1D Case: Basic bifurc. patterns

— stable

--- unstable

saddle-node	$b-y^2$		
Transcrit.	y^2+by		
Pitchfork	y^3+by		

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Bifurcations of nonlinear DS

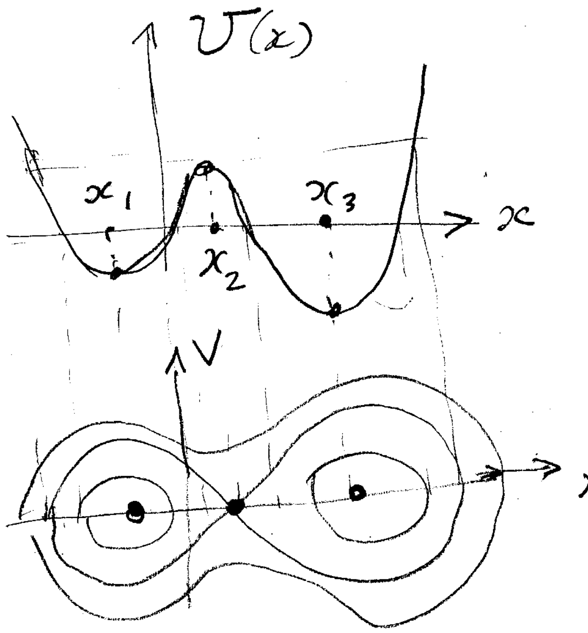
DS $\boxed{Y' = F(Y, a)}$ has equilibria $\{Y_1(a), Y_2(a) \dots\}$ whose type (source, sink, ... etc.) can change with parameter $\{a\}$, determined by Jacobian

$$A_i(a) = \begin{bmatrix} f_x(\dots; a) & f_y(\dots; a) \\ g_x(\dots; a) & g_y(\dots; a) \end{bmatrix}\Bigg|_{Y_i(a)}$$

Examples:

1) Friction in hamiltonian/mechanical systems $\begin{cases} \dot{x} = v \\ \dot{v} = -\alpha v - U'(x) \end{cases}$

$$A(\alpha) = \begin{bmatrix} 0 & 1 \\ -U''(x) & -\alpha \end{bmatrix}$$



	$(x_1, 0)$	x_2 0	...
	$\begin{bmatrix} 0 & 1 \\ -k_1 & -\alpha \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ k_2 & -\alpha \end{bmatrix}$	. .
$\alpha = 0$	center ↓	saddle ↓	
$\alpha > 0$	sink	saddle	

↑ Bifurcation!

Transcritical

Pred - prey w. logistic prey (varying C.C.)

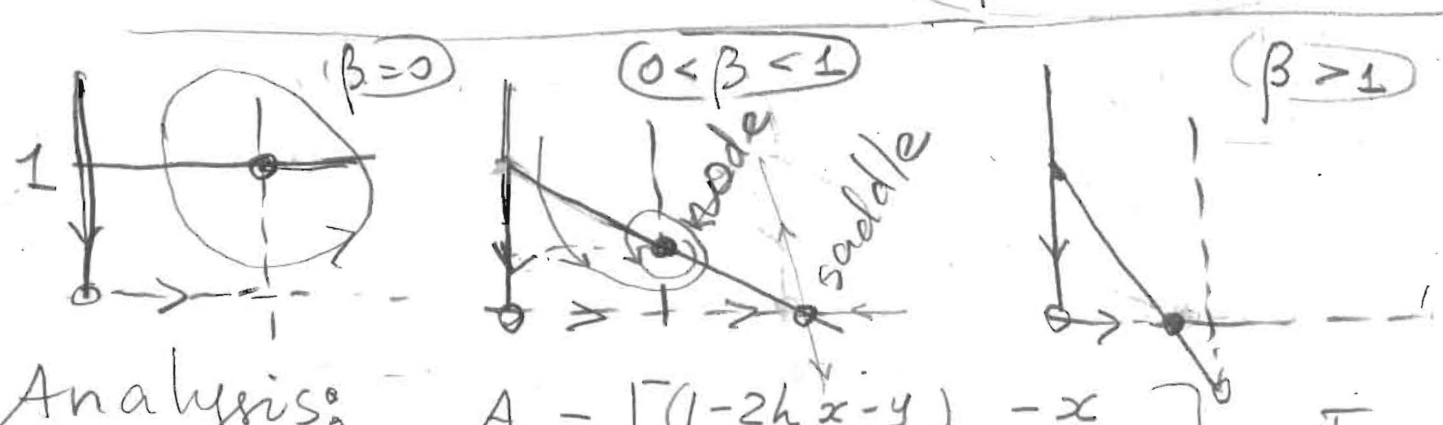
$$\begin{cases} \dot{x} = a\left(1 - \frac{x}{N}\right)x - bxy \\ \dot{y} = y(cx - d) \end{cases}$$ N - C.C. for x

1° Rescale: $x \to x/x_0$, $y \to y/y_0$, $t \to at$

Scales: $x_0 = d/c$, $y_0 = a/b$ ← standard VL equilibr.

$$\begin{cases} \frac{dx}{dt} = (1 - \beta x - y)x \\ \frac{dy}{dt} = \delta(x-1)y \end{cases}$$

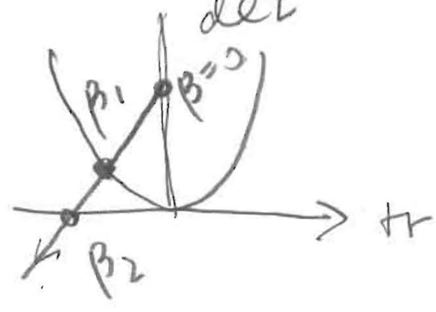
New parameters: $\delta = d/a$, $\beta = x_0/N$ $\quad 0 \le h < \infty$



Analysis: $A = \begin{bmatrix} (1 - 2hx - y) & -x \\ \delta y & \delta(x-1) \end{bmatrix}$ - Jac.

$(0,0)$	CC $(1/\beta, 0)$	$(1, 1-\beta)$
$\begin{bmatrix} 1 & 0 \\ 0 & -\delta \end{bmatrix}$	$\begin{bmatrix} -1 & -1/\beta \\ 0 & (1/\beta - 1) \end{bmatrix}$	$\begin{bmatrix} -\beta & -1 \\ \delta(1-\beta) & 0 \end{bmatrix}$

↪ $tr = -\beta$, $\det = \delta(1-\beta)$



Predator-Prey w. source

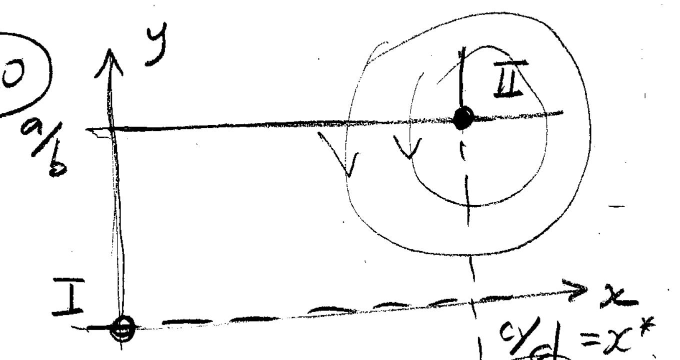
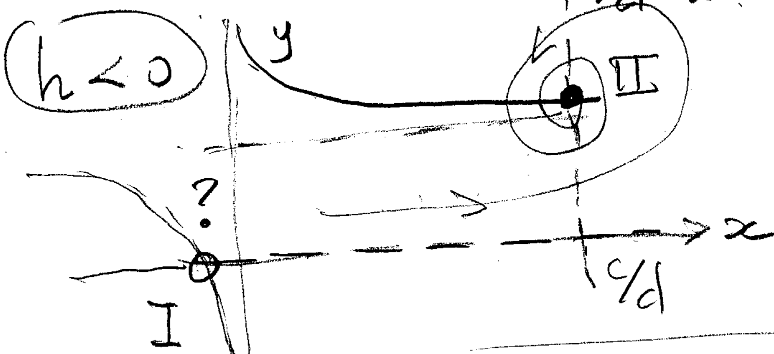
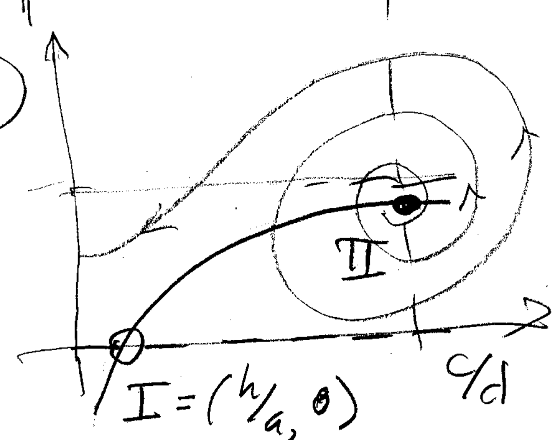
$$\begin{cases}\dot{x} = (a-by)x - h & \leftarrow \text{prey}\\ \dot{y} = (cx-d)y & \leftarrow \text{pred.}\end{cases}$$

$h > 0$ - harvest

$h < 0$ - source

$$\underline{\text{Jacobian}} = \begin{bmatrix}(a-by) & -bx\\ cy & (cx-d)\end{bmatrix}$$ independent of h!

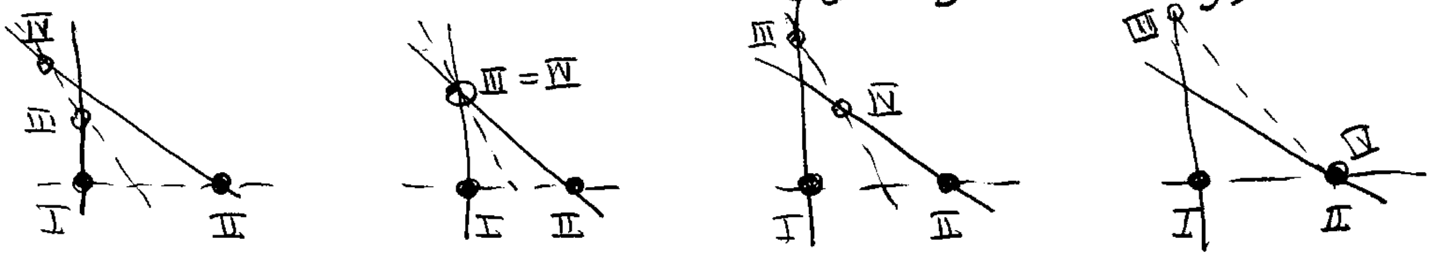
Equilibria change w. h

		I	II $=(x^*, y^*)$
$h=0$		$\begin{bmatrix}a & 0\\ 0 & -d\end{bmatrix}$ saddle	$\begin{bmatrix}0 & -bx^*\\ cy^* & 0\end{bmatrix}$ center
$h<0$		non-physic	$\begin{bmatrix}-\varepsilon & -bx^*\\ cy^* & 0\end{bmatrix}$ Sp. sink $\varepsilon = -h/x^*$
$h>0$		$\begin{bmatrix}a & bx^*\\ 0 & -d+\varepsilon'\end{bmatrix}$ $\varepsilon' = \frac{ch}{a}$ For small h, ε Saddle	$\begin{bmatrix}\varepsilon & -bx^*\\ cy^* & 0\end{bmatrix}$ $\varepsilon = h/x^*$ Sp. source

$$\boxed{y^* = \frac{a - hd/c}{b};\quad x^* = c/d}$$

$$p(\lambda) = \lambda^2 \mp \varepsilon\lambda + b$$

2) Competing species: $\begin{cases} \dot{x} = x(1-x-y) \\ \dot{y} = y(a-2x-y) \end{cases}$



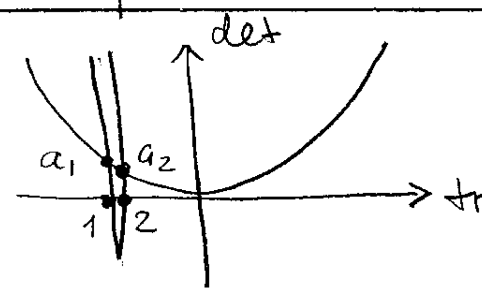
Equilibria III, IV and all type vary with a

$$\underline{A} = \begin{bmatrix} 1-2x-y & -x \\ -2y & a-2x-2y \end{bmatrix}$$

	$I = (0,0)$	$II = (1,0)$	$III = (0,a)$	$IV = (a-1, 2-a)$
$\underline{A}$	$\begin{bmatrix} 1 & \\ & a \end{bmatrix}$	$\begin{bmatrix} -1 & -1 \\ 0 & (a-2) \end{bmatrix}$	$\begin{bmatrix} 1-a & 0 \\ -2a & -a \end{bmatrix}$	$\begin{bmatrix} 1-a & 1-a \\ 2(a-2) & (a-2) \end{bmatrix}$
$\lambda_{1,2}$	$1, a$	$-1, (a-2)$	$1-a, -a$	$tr = -1$; $\det = (a-1)(a-2)$ $\lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \det}$
$a < a_1 = \frac{3}{2}(1-\sqrt{2})$	saddle	sink	source	sp. sink
$a_1 < a < 0$	— ıı —	— ıı —	— ıı —	sink
$0 < a < 1$	source	— ıı —	saddle	— ıı —
$1 < a < 2$	— ıı —	— ıı —	sink	saddle
$2 < a < a_2 = \frac{3}{2}(1+\sqrt{2})$	— ıı —	saddle	— ıı —	sink
$a_2 < a$	— ıı —	— ıı —	— ıı —	sp. sink

Bifurcation values for IV:

$$\det = (a-1)(a-2) = \begin{cases} \frac{1}{4} \Rightarrow a_{1,2} = \frac{3}{2}(1 \mp \sqrt{2}) \\ 0 \Rightarrow a_{3,4} = 1, 2 \end{cases}$$



a	I	II	III	IV
$(1-\sqrt{2})^{3/2}$	saddle	sink	source	sp. sink sink
0	saddle source	sink	source saddle	sink
1	source	sink	saddle sink	sink saddle
2	source	sink saddle	sink	saddle sink
$(1+\sqrt{2})^{3/2}$	source	saddle	sink	sink sp. sink

Limit cycles & Hopf bifurcation

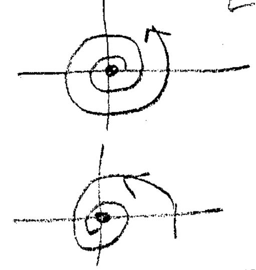
Take DS (1) $\begin{cases} \dot{x} = (a-r^2)x - \beta y \\ \dot{y} = (a-r^2)y + \beta x \end{cases}$ with fixed β

and parameter $a \gtrless 0$: $r^2 = x^2+y^2$ – radius

System (1) has equilibrium $(0,0)$ and

Jacobian: $A = \begin{bmatrix} (a-r^2-2x^2) & -(2xy+\beta) \\ (-2xy+\beta) & (a-r^2-2y^2) \end{bmatrix}$

So $A_0 = \begin{bmatrix} a & -\beta \\ \beta & a \end{bmatrix} \Rightarrow$ eigenvalues $\boxed{\lambda = a \pm i\beta}$

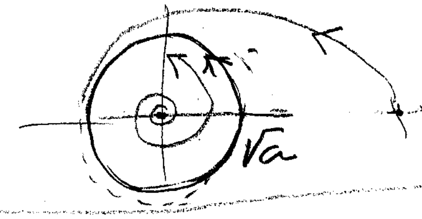


unstable

$\Rightarrow$ $\boxed{\begin{array}{ll} a>0 & \text{sp. source} \\ a<0 & \text{sp. sink} \end{array}}$

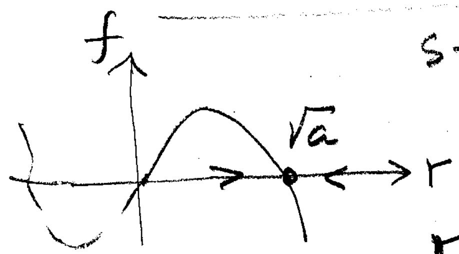
stable

For $a>0$ all solutions $Y(t) \to$ "limit cycle" = circle of radius $\sqrt{a}$



In polar coordinates: $x+iy = re^{i\theta}$

system (1) decouples: $\begin{cases} \dot{r} = (a-r^2)r = f(r) \\ \dot{\theta} = \beta \end{cases}$



stable equil. $r^* = \sqrt{a}$

separation: $\int \frac{dr}{(a-r^2)r} = t$; $\theta = \beta t$

$r(t) = \sqrt{a}\,\frac{e^{at}}{\sqrt{1+e^{2at}}} \longrightarrow \sqrt{a}$ as $t \to \infty$