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Method of multipliers for LDE

Multiplier f-n $\mu(t)$ of LDE: $\boxed{y' + a(t)y = b(t)}$

is given by $\boxed{\mu(t) = e^{\int_{t_0}^{t} a\,dt}}$

1) It solves homog. problem: $\boxed{\begin{array}{l} \mu' - a(t)\mu = 0 \\ \mu(t_0) = 1 \end{array}}$

$\boxed{y' + a y = 0} \Rightarrow \boxed{y(t) = \frac{c}{\mu(t)}}$

2) For inhomog. (source) problem

$$\underbrace{\mu\,|\,(y' + a\,y)}_{} = b \Rightarrow \boxed{(\mu y)' = \mu b}$$

GS: $\boxed{y(t) = \frac{1}{\mu(t)}\left[C + \int^{t}(\mu b)\,dt\right]}$

IVP: $\boxed{y(t) = \frac{1}{\mu(t)}\left[y_0 + \int_{t_0}^{t}(\mu b)\,dt\right]}$ for normal $\boxed{\mu(t_0) = 1}$

Case: $a = \text{const} \Rightarrow \mu(t) = e^{at}$

$$y(t) = e^{-at}\left[y_0 + \int_0^t e^{as} b(s)\,ds\right] = \underbrace{y_0 e^{-at}}_{y_h} + \underbrace{\int_0^t e^{-a(t-s)} b(s)\,ds}_{y_p}$$

$\boxed{y_p(t) = e^{-at} * b(t)}$ ← convolution

Examples of LDE problem: y'+ a y =b(t), with different sources b(t), and constant a

$b(t)$	Particular y_p	GS/IVP
i) const b_0	b_0/a (equil.)	$\frac{b_0}{a} + \left(y_0 - \frac{b_0}{a}\right)e^{-at}$ y_p
ii) polynom. $b_0 + b_1 t$	$\frac{b_0}{a} + \frac{b_1}{a}\left(t - \frac{1}{a}\right)$	$y_p + c\, e^{-at}$
iii) exponent. $b_0 e^{\beta t}$	$\frac{b_0}{\beta + a} e^{\beta t}$	$y_p + C\, e^{-at}$
iv) trig. $\cos \beta t$	$\frac{a \cos\beta t + \beta \sin\beta t}{a^2 + \beta^2}$	$y_p + C\, e^{-at}$

Examples of LDE problem: y'+ a y =b(t), with variable a(t). Multiplier method

$a(t)$	$\mu(t)$	GS/IVP
t	$e^{t^2/2}$	$e^{-t^2/2}\left(y_0 + \int_0^t e^{-s^2/2} b(s)\,ds\right)$
α/t	t^{α}	$t^{-\alpha}\left(c + \int_{t_0}^t s^{\alpha} b(s)\,ds\right)$
$\sin\beta t$	$e^{\frac{\cos\beta t}{\beta}}$	$e^{-\frac{\cos\beta t}{\beta}}\left(C + \int_0^t e^{\cos\beta s/\beta}\, b(s)\,ds\right)$

Multipliers for Bernoulli Eq-n

(1) $$\boxed{y' + a(t)\, y = b(t)\, y^n}$$ $\quad n \neq 1$

Use multiplier $\mu(t) = e^{\int_{t_0}^{t} a\, dt}$ $\Longrightarrow$

(2) $$\boxed{(\mu y)' = \frac{b(t)}{\mu^{n-1}} (\mu y)^n}$$ $\leftarrow$ separable DE for $z = \mu y$

Problem 1:

(a) Use separation to get GS of (1)

$$\boxed{y(t) = \frac{1}{\mu(t)} \left[C + (1-n) \int_{t_0}^{t} \frac{b}{\mu^{n-1}}\, dt \right]^{1/1-n}}$$

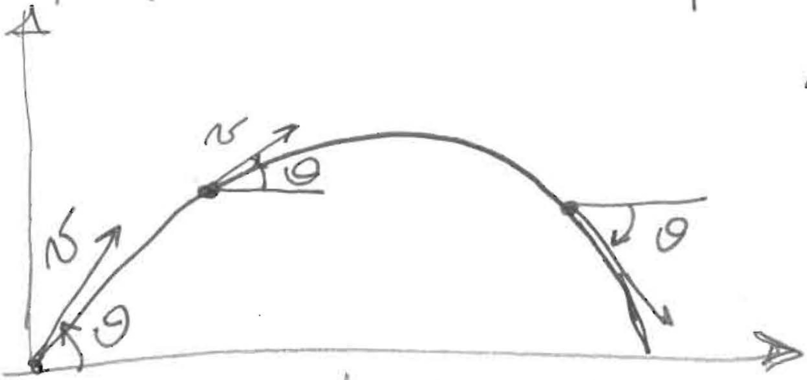
(b) Write IVP solution $y(t_0) = y_0$ assuming $\mu(t_0) = 1$

Problem 2: Solve & plot DEs

a) $y' + ay = by^3$; a, b - const $\left(\text{show: } y = \dfrac{e^{-at}}{\sqrt{C + \frac{b}{a} e^{-2at}}} \right)$

b) $y' + \frac{a}{t} y = y^3$

Problem 3: Projectile motion w. air resistance proportional to speed v.



Ask for f-n $v(\theta)$ -?

DE: $$\boxed{\frac{dv}{d\theta} = (\tan\theta) v + \frac{K}{\cos\theta} v^3}$$ $K = \frac{\alpha}{g}$ - $\frac{\text{friction}}{\text{gravity}}$

Solve DE analyticaly & plot f-ns $v(\theta)$

Problem 4: Logistic DE w. variable carrying capacity

$N = N(t)$; e.g. seasonal $N(t) = N_0(1 + b\cos\omega t)$

a) Solve by Bernoulli method and show that for const N it gives the same f-lae as separation

b) Derive f-la for variable $N(t)$ in terms of convolution integral $e^{-at} * \frac{1}{N(t)} = \int_0^t \frac{e^{-a(t-s)}}{N(s)} ds$ ..

$$\boxed{y(t) = \frac{1}{e^{-at}/y_0 + (e^{-at} * 1/N)}}$$

c) Study solutions numerically for periodic $N(t) = N_0(1 + b\cos\omega t)$ for a range $0 \le b \le 1$

Describe their behavior at large t, as b increases.

Is mean pop. size $\boxed{\overline{y(t)} = \frac{1}{T} \int_t^{t+T} y(s) ds \approx N_0}$ ← mean CC ?