Periodic force: modulation & resonance

$$\boxed{\ddot{x} + \frac{k}{m}x = f\cos\omega t;} \qquad \beta = \sqrt{k/m} \text{ - natural freq.}$$

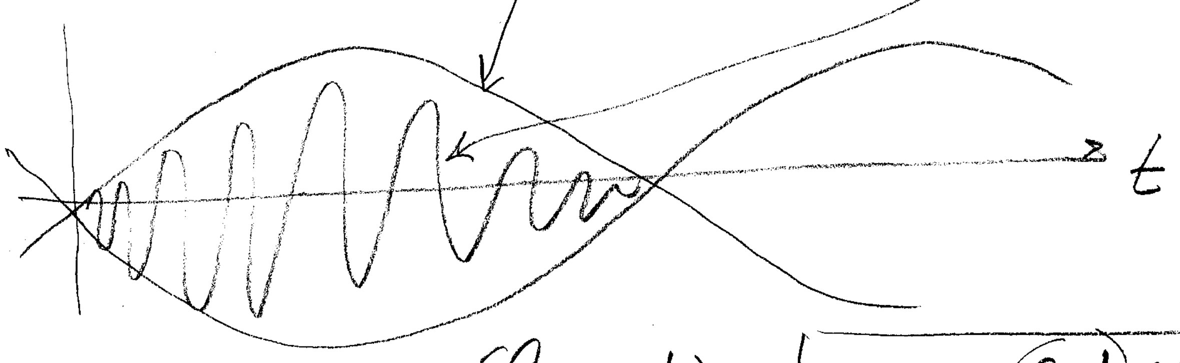
GS: $$x(t) = \underbrace{\frac{f\cos\omega t}{\beta^2-\omega^2}}_{\text{responce}} + \underbrace{c_1\cos\beta t + c_2\sin\beta t}_{\text{nat. oscill.}}$$

$\boxed{\omega \neq \beta}$

IVP: $\boxed{\begin{matrix} x(0)=0 \\ \dot{x}(0)=0 \end{matrix}} \Rightarrow$

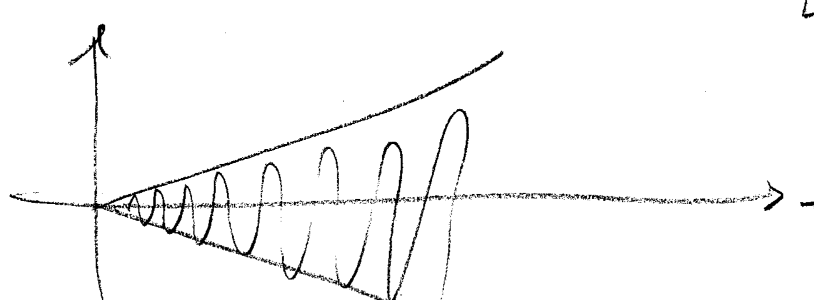
$$x(t) = f\,\frac{\cos\omega t - \cos\beta t}{\beta^2-\omega^2} = f\ \overset{\text{I}}{\boxed{\frac{\sin\left(\frac{\omega-\beta}{2}\right)t}{\left(\frac{\omega-\beta}{2}\right)}}}\ \overset{\text{II}}{\boxed{\frac{\sin\left(\frac{\omega+\beta}{2}\right)t}{(\omega+\beta)}}}$$

1° Case: $\omega \approx \beta$ - modulation



2° Case: $\omega = \beta$ (limit): $\boxed{x(t) = f\,t\,\frac{\sin\omega t}{\omega}}$

Resonance



Periodic force: damped oscillators

Linear oscill. with periodic force

$$m\ddot{x} + d\dot{x} + kx = f_0 \cos \omega t$$

(ω: frequency; f_0: amplitude)

Responce solution:

$$x_p(t) = f_0 \,\mathrm{Re}\left[\frac{e^{i\omega t}}{p(i\omega)}\right] = A\cos(\omega t - \phi)$$

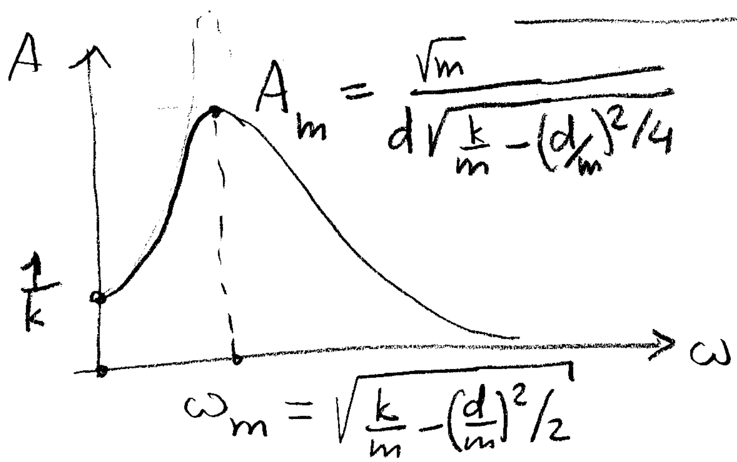
(A, ϕ: Responce amplit. & phase shift)

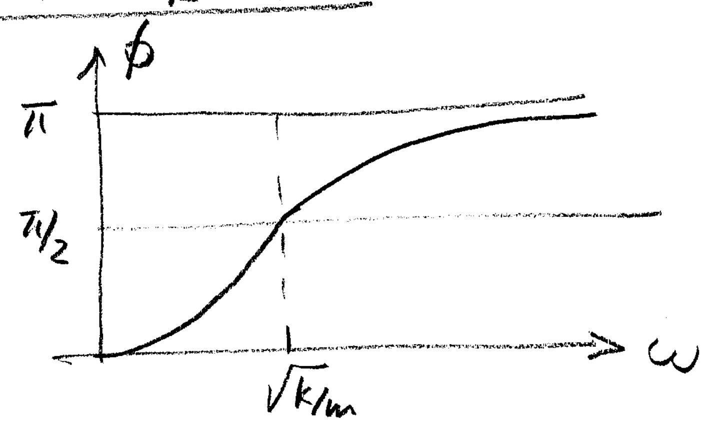
Charact. polynomial $p(\lambda) = m\lambda^2 + d\lambda + k$ gives

$$p(i\omega) = (k - m\omega^2) + i d\omega$$

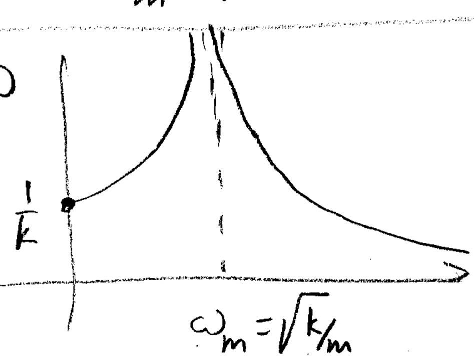
Amplitude: $A = \dfrac{1}{|p(i\omega)|} = \dfrac{1}{\sqrt{(k-m\omega^2)^2 + (d\omega)^2}}$

Phase-shift: $\phi = \tan^{-1}\left(d\omega / (k - m\omega^2)\right)$





As $d \to 0$



Example: $\ddot{x} + 2\dot{x} + 5x = \cos\omega t$

ω	$x_p(t) = \dfrac{\cos(\omega t - \phi)}{\sqrt{(5-\omega^2)^2 + (2\omega)^2}}$	$\phi = \tan^{-1}\left(\dfrac{2\omega}{5-\omega^2}\right)$
0	$1/5$	0
1	$\dfrac{\cos(t-\phi)}{\sqrt{4^2+2^2}} = \dfrac{\cos(t-\phi)}{2\sqrt{5}}$	$\tan^{-1}\left(\frac{1}{2}\right) = \cdots$
2	$\dfrac{\cos(2t-\phi)}{\sqrt{1^2+4^2}} = \dfrac{\cos(2t-\phi)}{\sqrt{5}}$	$\tan^{-1}(4) = \cdots$

Max responce:

$$\omega_m = \sqrt{5 - 2^2/2} = \sqrt{3}$$

$$A_m = \frac{1}{2\sqrt{5 - 2^2/4}} = \frac{1}{4}$$