Math224-228:

Modeling with Differential Equations

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A 1st order DE model of a continuous process/ system whose state is described by variable $y(t)$ has form: $\frac{dy}{dt} = f(y,t)$, where $f(y,t)$ - dynamic law, that determines the rate of change of $y(t)$. Solutions of such systems are functions of time $y(t;...)$, that also depend on other parameters and inputs (initial state, coefficients of the equation, etc.). We list a few examples and show typical solution curves as functions of t.

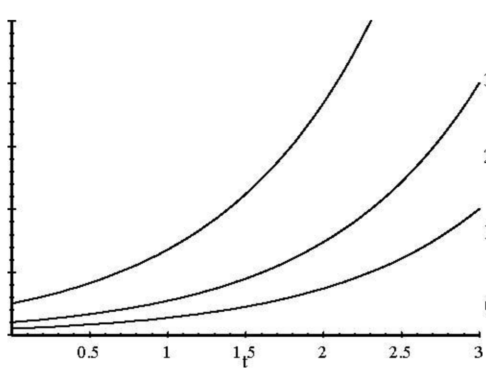
Examples:

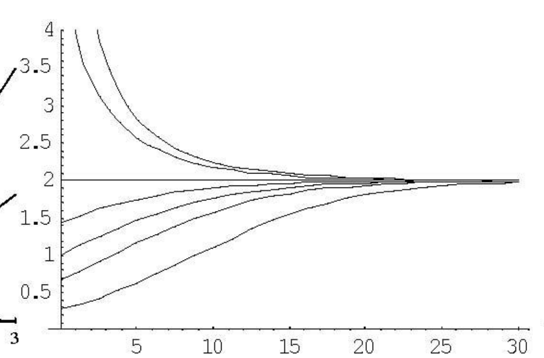
1. Population growth: $y(t)$-population size at time t

$$\frac{dy}{dt} = \text{"birth rate" - "death rate"}$$

i) Linear growth model: $y' = ky$, k =[1/time] -growth rate. Solutions $y(t) = Ce^{kt}$;

ii). Logistic growth: $y' = k(1 - y/N)y$; k - max growth rate, N -carrying capacity





2. Chemical reactions

1st order reaction: A→product, $y' = -ky$, y -concentration of A, k=[1/time] - decay rate. 2nd order reaction: A+B → product; $x(t), y(t)$- concentrations of A,B; k_2 = [1/"concentration" x "time"], obey a coupled DE system:

$$x' = -k_2 xy$$
$$y' = -k_2 xy$$

From mass conservation $x(t) - y(t) = a$ - constant, the system is reduced to a single DE for $x(t)$: $x' = -k_2 x(a+x)$.

3 Mechanical systems

based on Newton law: $ma = f$;

$$x(t) \text{ - position}$$
$$v(t) = \dot{x}(t) \text{ - velocity}$$
$$a(t) = \dot{v} = \ddot{x} \text{ - acceleration}$$

gives 2nd order DE for position $x(t)$: $m\ddot{x} = f(x, \dot{x}, t)$, or couple system (DS) for position-velocity $(x(t), v(t))$:

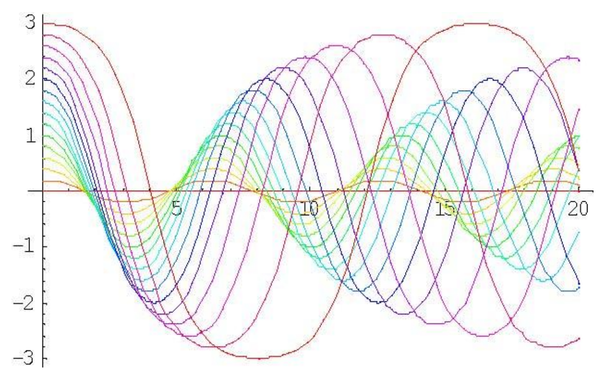
$$\dot{x} = v$$
$$\dot{v} = \frac{1}{m} f(x, v, t)$$

3.1 **Linear oscillator** (mass-spring): $m\ddot{x} = -kx$, 2nd order DE in x(t); m -mass, k -spring constant.

Solutions: $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$, $\omega = \sqrt{k/m}$ - frequency of oscillation.

3.2 **Pendulum**

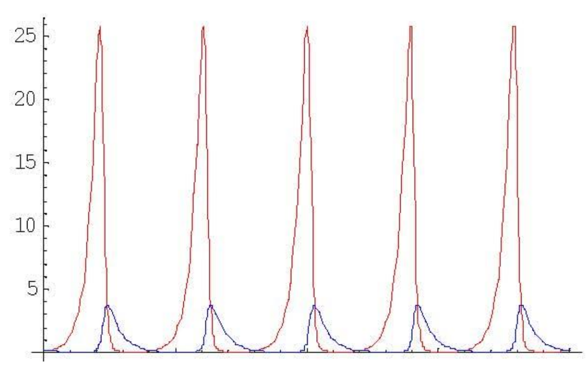
Pendulum of length *l* with position function measured by angle $\theta(t)$ from downward vertical obeys the Newton's equation: $l\ddot{\theta} = -g \sin\theta$, g = 9.8m/sec^2 - gravity acceleration. It looks similar to mass-spring, but has nonlinear "spring-force" $f = -g \sin\theta$.



4 Competition - cooperation

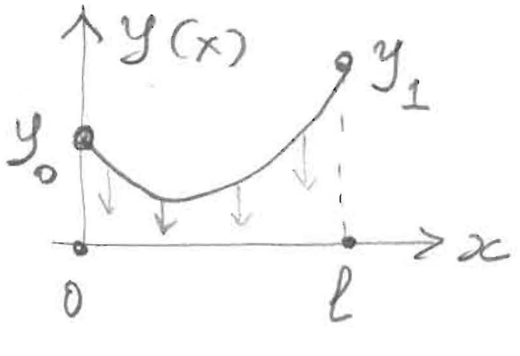
Two competing-cooperating species with linear growth law: $\begin{cases} x' = k_1 x \pm a_1 xy \\ y' = k_2 y \pm a_2 xy \end{cases}$, $x(t), y(t)$ -populations, k_1, k_2 - growth rates, a_1 , a_2 - interaction coefficients (find the dimensions of a_1, a_2). Similar model with logistic growth for each: $\begin{cases} x' = k_1 (1 - x/N_1)\, x \pm a_1 xy \\ y' = k_2 (1 - y/N_2)\, y \pm a_2 xy \end{cases}$.

Predator-prey: $\begin{cases} x' = k_1 x - a_1 xy & \text{-prey} \\ y' = a_2 xy - by & \text{-predator} \end{cases}$; a_1 - killing rate/{predator, prey, time}; a_2 - predator growth rate due to kill/{predator, prey, time}, b - attrition rate (for predator).



Models of Geometry & Statics

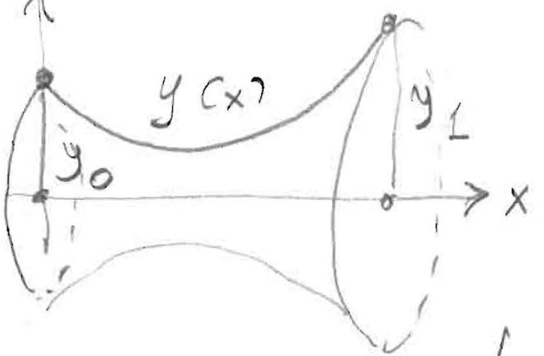
1. Heavy chain: find shape $y(x)$ - ?



$\Rightarrow$ $$y' = \sqrt{\left(\frac{y-a}{b}\right)^2 - 1}$$

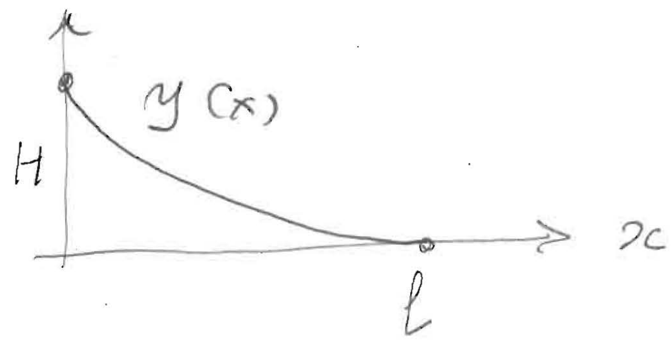
Parameters (a, b) determined from BC: $y(0) = y_0$; $y(l) = y_1$ and length $= L$

2. Minimal surface of revolution: $y(x)$



$$y' = \sqrt{\left(\frac{y}{c}\right)^2 - 1}$$ parameter c - ?

3. Fastest slope: $y(x)$ - ?



$$y' = \sqrt{\frac{c+y}{H-y}}$$ param. c - ?

US population: linear vs. logistic model

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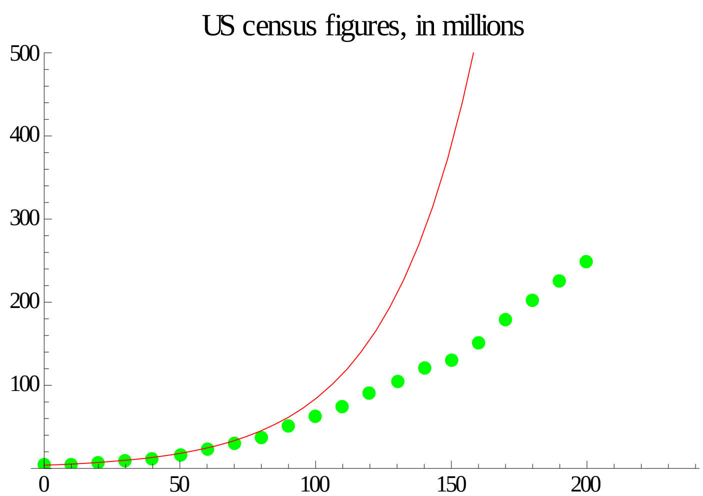
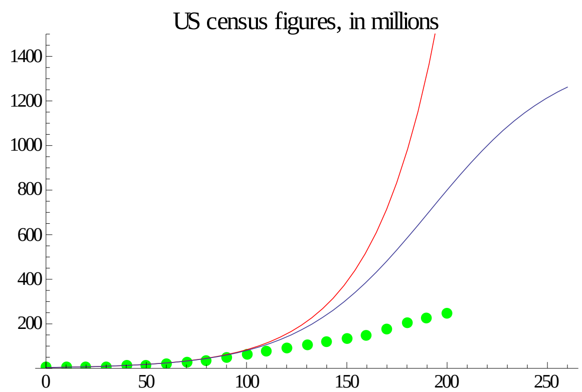
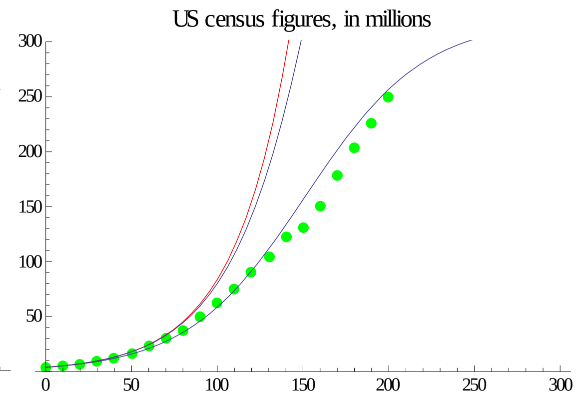
Year	Actual	Relative GR
0	3.9	
10	5.3	0.030673
20	7.2	0.0306374
30	9.6	0.0287682
40	12.	0.0223144
50	17.	0.0348307
60	23.	0.0302281
70	31.	0.0298493
80	38.	0.0203599
90	50.	0.0274437
100	62.	0.0215111
110	75.	0.0190354
120	91.	0.0193371
130	105.	0.0143101
140	122.	0.0150061
150	131.	0.00711763
160	151.	0.0142083
170	179.	0.0170106
180	203.	0.012582
190	226.	0.0107329
200	249.	0.00969179



Linear growth based on $\{y_0, y_1\}$





Logistic curve based on first 3 data points

Logistic curve based on all data points