Math. 224 Supplement: Mixing problems

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Mixing problem with the in/out rates: r_1, r_2 is described by the differential equation

$$q' + \frac{r_2}{V} q = \alpha_1 r_1 \tag{1}$$

where q denotes the amount of the chemical, α_1-the incoming concentration and $V = V_0 + (r_2 - r_1)\, t$ - volume.

0.1 Equal rates: $r_1 = r_2 = r$

Here V remains constant and equation (1) becomes

$$q' + \frac{r}{V} q = \alpha_1 r$$

We call its constant coefficients $\frac{r}{V} = a$ and $\alpha_1 r = b$ rewrite it as

$$q' + aq = b$$

and produce exact solution : $q(t) = \frac{b}{a} + \left(q_0 - \frac{b}{a}\right) e^{-at}$

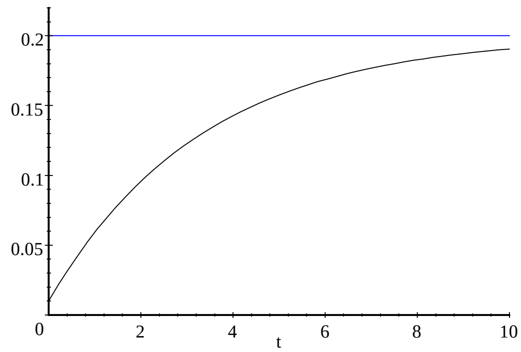
Specific example: $r_1 = r_2 = 2$; $\alpha_1 = .3$; $V = 10$ gives

$$q' + \frac{2}{10} q = \frac{6}{10}$$
$$q(0) = \frac{1}{10}$$

Exact solution is : $q(t) = 3 - \frac{29}{10} e^{-\frac{1}{5}t}$ and the corresponding concentration

$$\alpha(t) = \frac{q(t)}{V} = .3 - \frac{29}{100} e^{-\frac{1}{5}t}$$

clearly approaches the incoming (equilibrium) value $\alpha_1 = .3$



0.2 Different rates: $r_1 > r_2$

Here volume $V = V_0 + (r_1 - r_2)\, t$ changes (linearly) with t. Equation (1) is still linear,

$$q' + aq = b \tag{2}$$

but has variable coefficients $a\,(t) = \frac{r_2}{V(t)}$ and $b = \alpha_1 r_1$. The multiplier of (2) becomes

$$\mu\,(t) = e^{\int a dt} = V^{-\beta}$$

- a fractional power of the volume-function with exponential

$$\beta = \frac{r_2}{r_1 - r_2}$$

Its general solution gives

$$\begin{aligned} q\,(t) &= \frac{1}{\mu\,(t)}\left(C + \int \mu b\, dt\right) = CV^{-\beta} + \alpha_1 V \text{ - amount} \qquad (3) \\ \alpha\,(t) &= \frac{q}{V} = \alpha_1 + CV^{-\beta-1}\text{- concentration} \end{aligned}$$

Parameter C is related to the initial condition y_0 via

$$C = (q_0 - \alpha_1 V_0)\, V_0^{\beta}.$$

0.3 Specific examples:

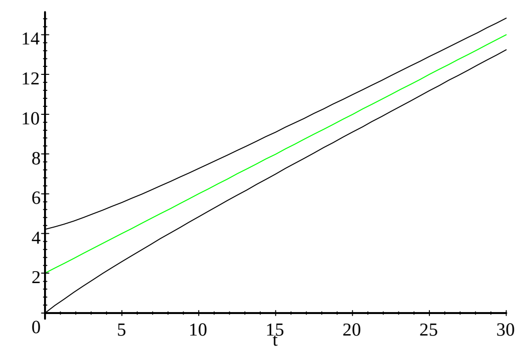
Example 1 *(Increasing volume):* $r_1 = 3; r_2 = 1$; $\alpha_1 = .2$; $V_0 = 10$. *Then* $V = 10 + 2t$, *and differential equation (1) becomes*

$$q' + \tfrac{1}{10+2t} q = \tfrac{6}{10}$$
$$q(0) = q_0$$

Here exponent $\beta = \frac{1}{2}$, *multiplier* $\mu = (10 + 2t)^{1/2}$, $C = (q_0 - 2)\sqrt{10}$, *and solutions*

$$q(t) = C(10 + 2t)^{-1/2} + .2(10 + 2t)$$

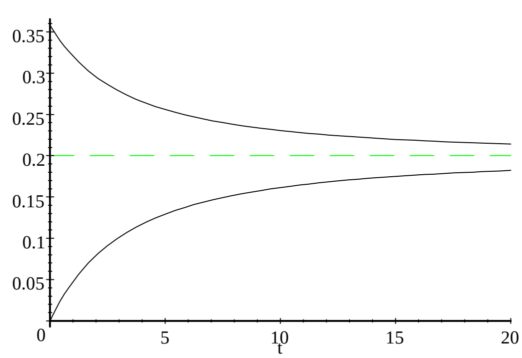
with different (initial) C



are asymptotically linear. But the corresponding concentrations

$$\alpha(t) = \frac{q}{V} = .2 + C(10 + 2t)^{-3/2}$$

once again approach limiting value $.2 = \alpha_1$- *the incoming one*



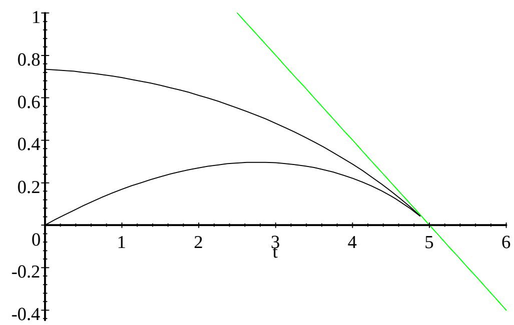
Example 2 *(Decreasing volume):* $r_1 = 1; r_2 = 3$; $\alpha_1 = .2$; $V_0 = 10$. *Then* $V = 10 - 2t$, *and differential equation (1) becomes*

$$q' + \tfrac{3}{10-2t}q = .2$$
$$q(0) = q_0$$

Here exponent $\beta = -\frac{3}{2}$, *multiplier* $\mu = (10+2t)^{-3/2}$, $C = (q_0 - 2)\,10^{-3/2}$, *and solutions*

$$q(t) = C(10-2t)^{3/2} + .2(10-2t)$$

with different (initial) q



all expire in finite time $t = 5$, *when volume V become* 0. *But the corresponding*

$$\alpha(t) = \frac{q}{V} = .2 + C(10-2t)^{1/2}$$

still approach limiting value $.2 = \alpha_1$ *of the incoming concentration*