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Reconstructing matrix from eigendata

E-V problem: given A find $\left\{\begin{matrix} \lambda_1 & \lambda_2 & \dots \\ X_1 & X_2 & \dots \end{matrix}\right\}$?

Inverse E-V problem: given $\left\{\begin{matrix} \lambda_1 & \lambda_2 & \dots \\ X_1 & X_2 & \dots \end{matrix}\right\}$ find $\underline{A}$?

Solution: $A X_1 = \lambda_1 X_2$; $A X_2 = \lambda_2 X_2$; ...

Define matrix of eigenvectors: $U = [X_1 \; X_2 \dots]$

Then $AU = U \begin{bmatrix} \lambda_1 & 0 & \\ 0 & \lambda_2 & \\ & & \ddots \end{bmatrix}$ ← diagonal

$$\Rightarrow \boxed{\underline{A} = U \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} U^{-1}}$$ Reconstruction f-la

Example: Eigendata $\left\{ \begin{array}{c|c} \lambda & \mu \\ \binom{1}{a} & \binom{1}{b} \end{array} \right\}$. Define $U = \begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix}$

Compute $U^{-1} = \frac{1}{b-a}\begin{bmatrix} b & -1 \\ -a & 1 \end{bmatrix} \Rightarrow \underline{A} = \begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix}\begin{bmatrix} \lambda & \\ & \mu \end{bmatrix}\begin{bmatrix} b & -1 \\ -a & 1 \end{bmatrix} / (b-a)$

$$\Rightarrow \boxed{A = \frac{1}{b-a}\left[\begin{array}{c|c} \lambda b - \mu a & \mu - \lambda \\ \hline ab(\lambda-\mu) & \mu b - \lambda a \end{array}\right]} \qquad (*)$$

Problem: (i) Show that for any complex eigendata $\left\{\begin{matrix} \lambda = \alpha \pm i\beta \\ X = \binom{1}{x \pm iy} \end{matrix}\right\}$ matrix $\underline{A}$ reconstructed via $(*)$ is real.

(ii) Compute $\underline{A}$ in terms of α, β, x, y

(iii) Apply to eigendata: $\left\{\begin{matrix} \lambda = -1+2i \\ X = \binom{1}{2-i} \end{matrix}\right\}$

Matrix Exponential

1) $e^{tA} = \sum_{0}^{\infty} \frac{t^n}{n!} A^n$ – convergent series, requires all powers $\{A^n\}$ and infinite summation

2) For diagonal $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 & \\ & & \ddots \end{bmatrix}$; $A^n = \begin{bmatrix} \lambda_1^n & & \\ & \lambda_2^n & \\ & & \ddots \end{bmatrix}$

So $e^{tA} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \end{bmatrix}$ – also diagonal.

3) Any (generic) A is diagonalized by the matrix of eigenvectors: $\underline{A} = U \underline{\Lambda} U^{-1}$

$\Rightarrow$ $A^n = U \Lambda^n U^{-1}$ and

$$\boxed{e^{tA} = U \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \end{bmatrix} U^{-1}}$$

Example: $A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$ has char. polyn. $p = \lambda^2 + a\lambda + b$

with eigendata: $\begin{array}{c|c} \lambda_1 & \lambda_2 \\ \hline \binom{1}{\lambda_1} & \binom{1}{\lambda_2} \end{array}$ $\Rightarrow$ $U = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$; $U^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$

$$\Rightarrow \quad e^{tA} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \frac{1}{(\lambda_2 - \lambda_1)} \begin{bmatrix} \lambda_2 & -1 \\ -1 & 1 \end{bmatrix} = \dots$$

Problem 2: (i) Compute e^{tA} in example

(ii) Apply to $\underline{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$; $C = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$

4) Solutions of linear ODS (like ODE)

(i) $\begin{cases} \underline{Y}' = A\underline{Y} \\ \underline{Y}(0) = Y_0 \end{cases}$ $\Rightarrow$ $\boxed{\underline{Y}(t) = e^{tA} \underline{Y}_0}$

(ii) $\underline{Y}' = A\underline{Y} + F(t)$ $\Rightarrow$ $\boxed{Y(t) = \int_0^t e^{A(t-s)} F(s)\,ds}$