M224
D. GURARIE

# Equilibria & linearization

For nonlinear autonomous system: $\Upsilon' = F(\Upsilon)$

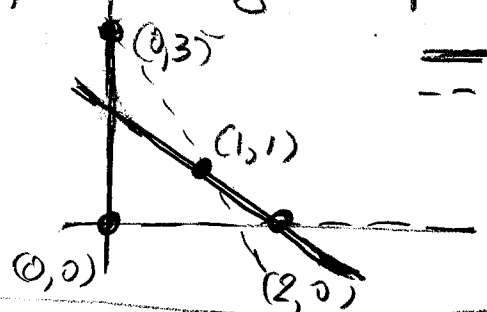
$\Upsilon = \begin{pmatrix} x \\ y \end{pmatrix}$ ; $F = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$ compute equilibria

$\boxed{F(\Upsilon) = \vec{0}} \Rightarrow \Upsilon_0, \Upsilon_1, \ldots$ and linearize $F$ at each one

$\Upsilon(t) \approx \Upsilon_0 + U(t)$, where $\boxed{\dot{U} = AU}$ with Jacobian matrix $A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Big|_{\Upsilon_0}$ ← approximate linear DS

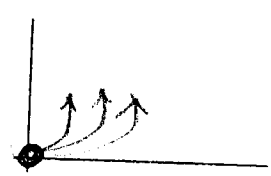
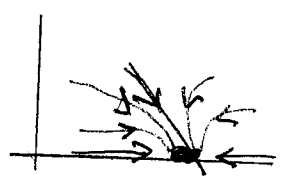
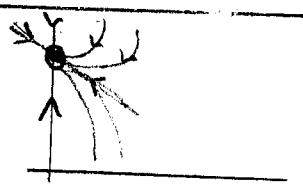
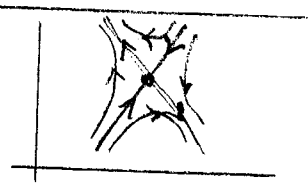
Example: competing species

$$\begin{cases} \dot{x} = 2x(1 - x/2) - xy \\ \dot{y} = 3y(1 - y/3) - 2xy \end{cases}$$




═══ x-nul
--- y-nul

Jacobian: $\boxed{F' = \begin{bmatrix} (2-2x-y) & -x \\ -2y & (3-2x-2y) \end{bmatrix}}$

| Equil. | $A$ | eigens | Approx. $\Upsilon \approx$ | Stab. type | Phase-plot |
|---|---|---|---|---|---|
| (0,0) | $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ | $2 \quad 3$<br>$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | $c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | source /unst. |  |
| (2,0) | $\begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix}$ | $-2 \quad -1$<br>$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ 0 \end{pmatrix} + c_1 e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ | sink /stab. |  |
| (0,3) | $\begin{bmatrix} -1 & 0 \\ -6 & -3 \end{bmatrix}$ | $-1 \quad -3$<br>$\begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 3 \end{pmatrix} + c_1 e^{-t}\begin{pmatrix} -1 \\ 3 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | sink /stab |  |
| (1,1) | $\begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$ | $-(1+\sqrt{2}) \quad -1+\sqrt{2}$<br>$\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \quad \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_1 e^{-(1+\sqrt{2})t}\begin{pmatrix} \sqrt{2} \\ \cdot \end{pmatrix} + c_2 e^{(\sqrt{2}-1)t}\begin{pmatrix} -\sqrt{2} \\ \cdot \end{pmatrix}$ | Saddle |  |