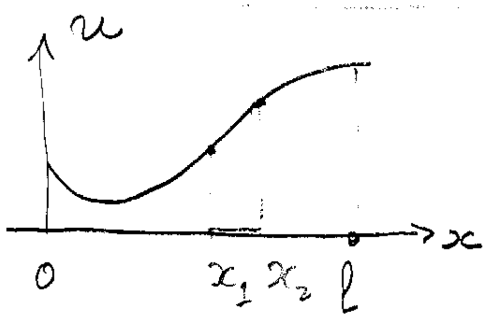
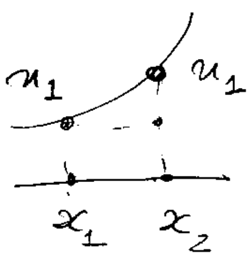
M224 Heat diffusion (Fourier law) and Boundary value problems.



$u(x,t)$ - temp. distribution in a rod of length l

1) Heat (internal energy): $\boxed{Q = \underbrace{\rho}_{\text{specific heat}} u}$

2) Heat flux (from "hot x_2" to "cold x_1")



$$\boxed{\frac{dQ}{dt} = \alpha \frac{\Delta u}{\Delta x} \longrightarrow \alpha u_x}$$

(Fourier)

α – coeff. of heat conductivity

Total heat on $[x_1\, x_2]$: $Q = \int_{x_1}^{x_2} \rho u \, d\xi$;

Balance law:

$$\frac{dQ}{dt} = \underbrace{\alpha u_x \Big|_{x_1}^{x_2}}_{\text{heat flux across boundary } \{x_1\, x_2\}} - \underbrace{\int_{x_1}^{x_2} \beta u \, d\xi}_{\text{heat radiation over entire length}} + \overbrace{\int_{x_1}^{x_2} F \, d\xi}^{\text{sources}}$$

$$\int_{x_1}^{x_2} \rho u_t \, d\xi = \int_{x_1}^{x_2} \left[(\alpha u_x)_x - \beta u + F \right] d\xi \leftarrow$$

Balance integral relation for any $[x_1\, x_2]$

$\Longrightarrow$ Heat-diffusion eq-n (pde)

$$\boxed{\rho u_t = \alpha u_{xx} - \beta u + F}$$

diffusion radiation source

Boundary conditions:

(i) fixed temp: $u\big|_{x=0} = T_1$; $u\big|_{x=\ell} = T_2$

(ii) fixed flux: $u_x\big|_{x=0} = \ldots$; $u_x\big|_{x=\ldots} = \ldots$

(iii) insulated (zero flux): $u_x\big|_{x=0} = 0$

Stationary (time indep.) temp. equilibria

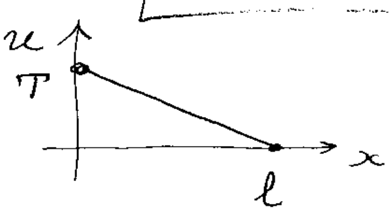
$$(*)\begin{cases} \alpha u_{xx} - \beta u = 0 & \text{on } [0, \ell] \\ u\big|_{x=0} = T \; ; \quad u\big|_{\ell} = 0 \; ; \end{cases}$$

2-point Boundary Value Problem (BVP)

Solutions of BVP (boundary value problem) *

1^0 $\beta = 0$: (no heat radiation)

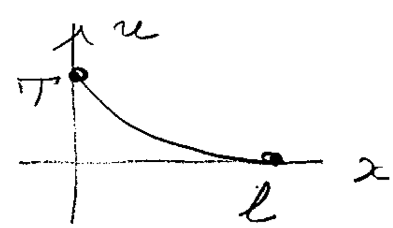
$$\begin{cases} u_{xx} = 0 \\ u\big|_0 = T \; ; \; u\big|_\ell = 0 \end{cases} \Rightarrow \quad u = c_1 + c_2 x \Rightarrow \boxed{u = T(1 - x/\ell)}$$



2^0 $\beta \neq 0$

$$\begin{cases} u_{xx} - \underbrace{\frac{\beta}{\alpha}}_{m^2} u = 0 \\ u\big|_0 = T ; \; u\big|_\ell = 0 \end{cases} \Rightarrow \quad u = \begin{cases} c_1 e^{mx} + c_2 e^{-mx} \\ c_1 \,\mathrm{ch}\, mx + c_2 \,\mathrm{sh}\, mx \end{cases} \Rightarrow$$

$$\boxed{u = T \, \frac{\mathrm{sh}\, m(\ell - x)}{\mathrm{sh}\, m\ell}}$$



Problems:

1. Derive sinh-solution in case 2°

2. Find equilibrium temp. distribution for radiating rod $u_{xx} - m^2 u = 0$ for BVP $u\big|_{x=0} = T$; $u_x\big|_{x=\ell} = 0$ (insulated). Plot solution

3. a) Find equilibrium temp. distribution for uniform heat source: $\alpha u_{xx} + F = 0$ on $[0, \ell]$ with insulated ends $u_x\big|_{x=0,\ell} = 0$. Plot solution (Hint: guess particular solution $u_p \overset{?}{=} a x^2$ and adjust it by homogeneous solution $u_h = c_1 + c_2 x$)

   b) The same problem for radiating rod:
   u''- 4 u = -F; u(0)=0; u(1)=0