Free fall and flight withy friction: parachute jump and volleyball serve.

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A body moving in a medium (gas, fluid, etc.) feels its resistance as a drag/friction force. This force usually depends on speed v of moving body, either linearly $F_d = -\alpha \mathbf{v}$ (for low speed motion), or at high speed one could get more complicated nonlinear (e.g. quadratic) force, $F_d = -\alpha\left(|\mathbf{v}|\right)\mathbf{v}$ - the so call Stokes drag. The Newton equations of motion with friction take on the form

$$\dot{\mathbf{x}} = \mathbf{v};\ \dot{\mathbf{v}} = \frac{1}{m}\mathbf{F} - \alpha\left(v\right)\mathbf{v} \tag{1}$$

where $\mathbf{x}$ - position, $\mathbf{v}$ - velocity, m - mass, $\mathbf{F} = \mathbf{F}\left(x\right)$ - external force (e.g. gravity), $\alpha\left(v\right)$ - suitable friction coefficient.

1 1D vertical motion

In 1D case, e.g. free (vertical) fall in the force of gravity (1) becomes a coupled system of 2 equations for height y and velocity v,

$$\dot{y} = v; \quad \dot{v} = -g - \alpha\left(v\right)v \tag{2}$$

Furthermore, for constant gravity g the second (velocity) equation is uncoupled from first. We can solve it directly for $v\left(t\right)$, and then integrate in t to get height-function

$$y\left(t\right) = y_0 + \int_0^t v\left(\tau\right)d\tau. \tag{3}$$

1.1 Linear friction.

A linear equation $\dot{v} = -g - \alpha v$ with initial value $v\left(0\right) = v_0$ has exact solution for v, and y (via integration (3))

$$\begin{aligned} v\left(t\right) &= -\frac{g}{\alpha} + \left(v_0 + \frac{g}{\alpha}\right)e^{-\alpha t} \\ y\left(t\right) &= y_0 + \left(v_0 + \frac{g}{\alpha}\right)\frac{1 - e^{-\alpha t}}{\alpha} - \frac{g}{\alpha}t \end{aligned} \tag{4}$$

Problem 1 *(i) Derive (4).*
(ii) Show that limit $\alpha \to 0$ *of (4) gives a frictionless free-fall solution:* $v(t) = v_0 - gt$; $y(t) = y_0 + v_0 t - \frac{gt^2}{2}$.

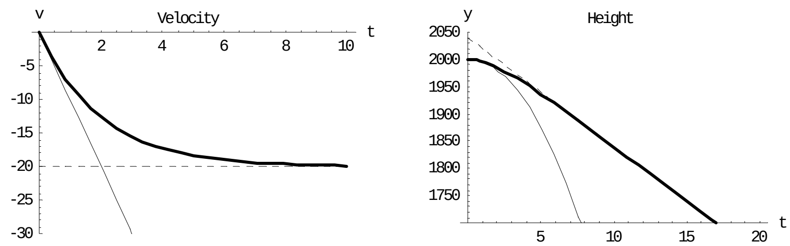

Figure 1: Velocity and height with linear friction (thick) vs. frictionless case (thin)

Note that $v(t)$ in (4) approaches finite limiting value $v^* = -\frac{g}{\alpha}$ (terminal velocity), unlike friction-less free fall where v grows linearly with time (plot above).

1.2 Parachute jump

Friction coefficient depends on geometric shape of moving object. Clearly, small α give large terminal velocity $-\frac{g}{\alpha}$ of free fall.

A parachute jump could be modeled by the velocity equation $\dot{v} = -g - \alpha v$, with step-function

$$\alpha(t) = \begin{cases} \alpha_1 & t < t_0 \\ \alpha_2 & t > t_0 \end{cases}$$

where α_1, α_2 are friction coefficients of the free fall and parachute fall (clearly $\alpha_1 \ll \alpha_2$), and t_0 - moment the parachute is opened. The corresponding solution is also given by a two-case function, they correspond to a two-step procedure: (i) solution $\{y_1(t), v_1(t)\}$ of free fall on time interval $[0, t_0]$ with coefficient α_1 and (ii) solution $\{y_2(t), v_2(t)\}$ of parachute fall for $t > t_0$ with coefficient α_2. Two IVP

$$(i) \begin{cases} \dot{v} = -g - \alpha_1 v & 0 < t < t_0 \\ v(0) = v_0 \end{cases} ; \quad (ii) \begin{cases} \dot{v} = -g - \alpha_2 v & t_0 < t \\ v(t_0) = v_1 \end{cases} \tag{5}$$

give two solutions

$$(i)\left\{\begin{array}{c} v_1(t) = -\frac{g}{\alpha_1} + \left(v_0 + \frac{g}{\alpha_1}\right) e^{-\alpha_1 t} \\ y_1(t) = y_0 + \left(v_0 + \frac{g}{\alpha_1}\right) \frac{1-e^{-\alpha_1 t}}{\alpha_1} - \frac{g}{\alpha_1} t \end{array}\right. ;\, t < t_0 \tag{6}$$

$$(ii)\left\{\begin{array}{c} v_2(t) = -\frac{g}{\alpha_2} + \left(v_1^* + \frac{g}{\alpha_2}\right) e^{-\alpha_2 (t-t_0)} \\ y_2(t) = y_1^* + \left(v_1^* + \frac{g}{\alpha_2}\right) \frac{1-e^{-\alpha_2 (t-t_0)}}{\alpha_2} - \frac{g}{\alpha_2}(t - t_0) \end{array}\right. ;\, t > t_0$$

Initial values of problem (ii) are given by the terminal of (i), $v_1 = v_1(t_0)$; $y_1, = y_1(t_0)$. Plot below show such two-step solution for $\alpha_1 = .5$, $\alpha_2 = 2.5$, when the parachute is open at $t_0 = 30s$.

1.3 Quadratic friction.

Here friction coefficient is given by $\alpha|v|$, so velocity equation becomes nonlinear, $\dot{v} = -g - \alpha|v|v = \left\{\begin{array}{ll} -g - \alpha v^2 & v > 0 \\ -g + \alpha v^2 & v < 0 \end{array}\right.$. It can by solved by separation,

$$\int \frac{dv}{g \pm \alpha v^2} = t - t_0$$

Hence solution

$$v(t) = -\sqrt{\frac{g}{\alpha}} \left\{\begin{array}{ll} \tan\left[\sqrt{g\alpha}(t - t_0)\right] & v > 0 \\ \tanh\left[\sqrt{g\alpha}(t - t_0)\right] & v < 0 \end{array}\right.$$

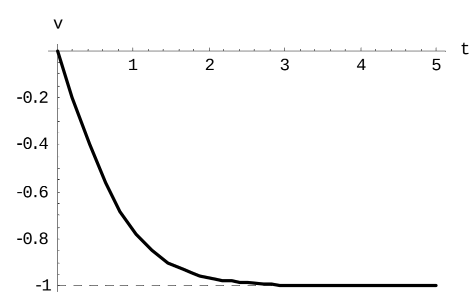

Figure 2: Quadratic friction velocity.

It still approaches limiting value, but the terminal velocity is now lower, $v^* = -\sqrt{g/\alpha}$.

2 2D motion: volleyball serve

In 2D case position $\mathbf{x} = (x, y)$ and velocity $\mathbf{v} = (u, w)$ are vectors. The velocity equations (1) are still uncoupled from $\mathbf{x}$ -equations. But this time velocity equation is itself a coupled system, linear (i) or nonlinear (ii)

$$(i)\begin{cases}\dot{u}=-\alpha u\\ \dot{w}=-g-\alpha w\end{cases};\quad (ii)\begin{cases}\dot{u}=-\alpha\left(\sqrt{u^2+w^2}\right)u\\ \dot{w}=-g-\alpha\left(\sqrt{u^2+w^2}\right)w\end{cases}$$

System (i) decouples and easily solved for u and w. Then trajectory $\{x(t), y(t)\}$ is computed from (u,w). Plot below demonstrates the effect of friction on heights and trajectories, that start with same initial value $(x_0, y_0, u_0, w_0) = (0, 1.5, 2, 3)$

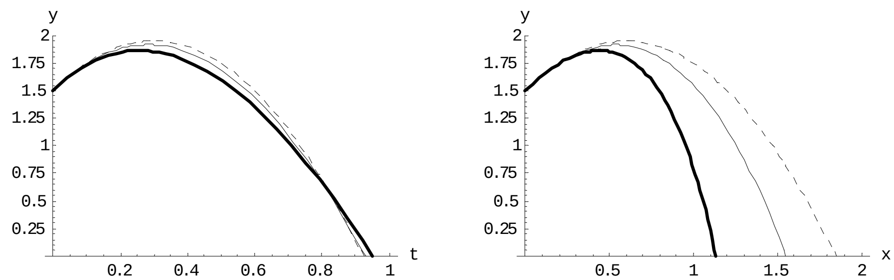

Figure 3: Height $y(t)$ (left) and trajectories (right) of the voleyball serve for 3 values of friction: $\alpha = 1.2$ (thick), $\alpha = .4$ (thin), $\alpha = 0$ (dashed).

2.1 Nonlinear friction.

System (ii) is coupled and nonlinear. It is convenient to write it in polar coordinates: $\mathbf{v} = v(\cos\theta, \sin\theta)$ - speed and inclination angle θ (or complex form $\mathbf{v} = ve^{i\theta}$). whence we get

$$\begin{cases}\dot{v}=-g\sin\theta-\alpha v^2\\ \dot{\theta}=-\frac{g\cos\theta}{v}\end{cases}\tag{7}$$

The new system is still coupled and there is no close form analytic solutions $v(t), \theta(t)$ in variable t. But (7) could be reduced to a single ODE, as any 2D autonomous system:

$$(a)\begin{cases}\dot{x}=f(x,y)\\ \dot{y}=g(x,y)\end{cases}\Rightarrow(b)\,\frac{dy}{dx}=\frac{g(x,y)}{f(x,y)}$$

Solutions of the latter (b), $\{y(x,C)\}$ describe trajectories of (a).

For system (7) the ODE turns into so called *Bernoulli equation*

$$\frac{dv}{d\theta}-\tan\theta v=-Kv^3\tag{8}$$

with coefficient $K = \frac{\alpha}{g}$.

A general nonlinear Bernoulli equation $y' - ay = by^n$ $(n \neq 1)$ with coefficients $a(x), b(x)$ could be transformed into a linear equation via change of variable: $y \to w = y^{1-n}$.

Problem 2 *Show that Bernoulli w satisfies a linear ODE $w' + (n-1)aw = (1-n)b$, and write its multiplier solution.*

The corresponding change for (8), $w = 1/v^2$, gives linear ODE: $w' + 2\tan\theta w = -\frac{2K}{\cos\theta}$. Its multiplier solution $\left(\frac{w}{\cos^2}\right)' = -\frac{2K}{\cos^3}$ could be integrated in closed form to get

$$w = K\cos^2\left(C - \frac{\sin}{\cos^2} - \ln\left|\frac{1+\sin}{\cos}\right|\right) \tag{9}$$

Hence the general solution of (8) with constant of integration C

$$v(\theta, C) = \frac{1}{\sqrt{K}\cos\theta\left(C - \frac{\sin\theta}{\cos^2\theta} - \ln\left|\frac{1+\sin\theta}{\cos\theta}\right|\right)} \tag{10}$$

We plot a few solutions v (below) for several values C on the range: $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$,

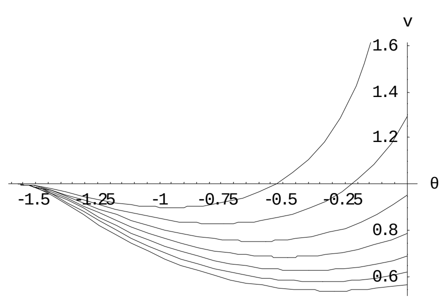

Figure 4: Solutions of (8) for coefficient $K = 1$.

Note that all $v(\theta, C)$ have as above limiting value (as $\theta \to -\pi/2$) - the terminal speed $v^* = 1/\sqrt{K} = \sqrt{\frac{g}{\alpha}}$.

Problem 3 *(i)Derive solutions (9) and (10) (Hint: $\int \frac{1}{\cos^3} = \frac{1}{2}\left[\frac{\sin}{\cos^2} + \ln\left(\frac{1+\sin}{\cos}\right)\right]$).*
(ii) Derive the limiting velocity.
(iii) Solve numerically for v, θ, x, y as functions of t, and plot the volleyball trajectories in the (x, y) plane for different values of friction coefficient α.