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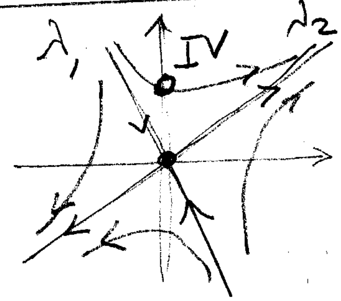
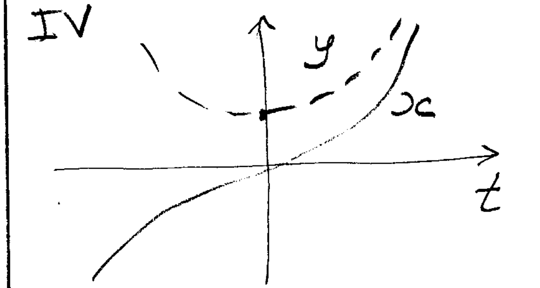
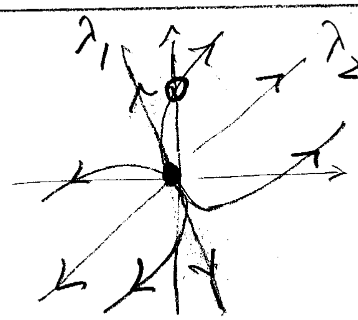
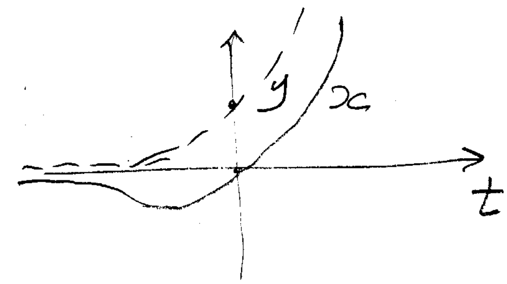
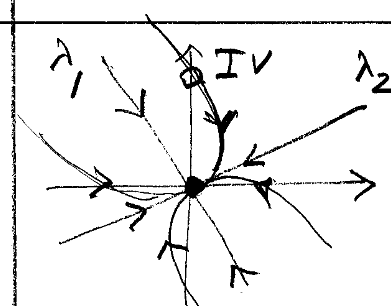
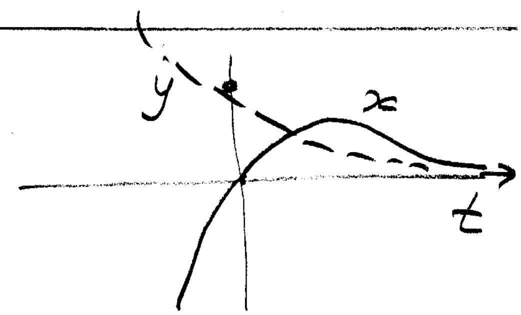
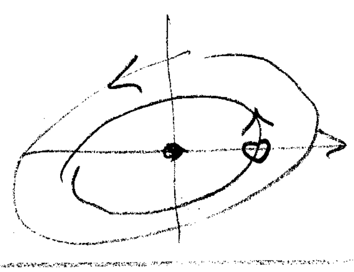
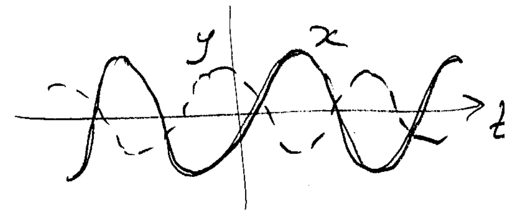
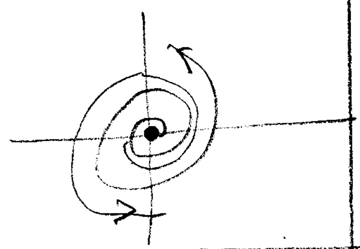
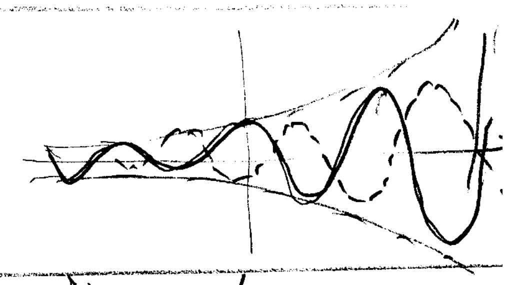
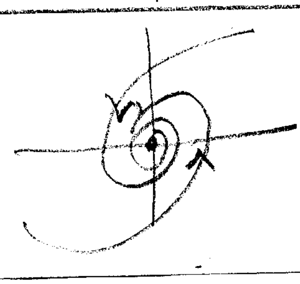
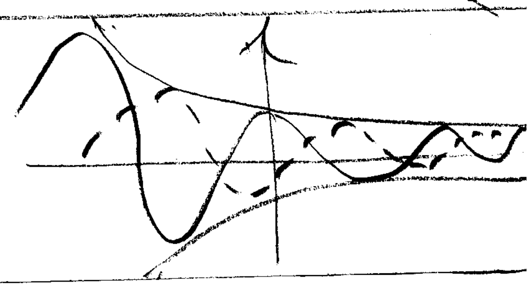
D. Gurarie

Equilibria types for 2D LDS.

$\lambda_{1,2}$	Type	Phase-plane	Solution curves
$\lambda_1 < 0 < \lambda_2$	saddle (unstab)		
$0 < \lambda_1 < \lambda_2$	source (unstab)		
$\lambda_1 < \lambda_2 < 0$	sink (stable)		
$Re\,\lambda = 0$ $\lambda = \pm i\beta$	center (neutr.)		
$Re\,\lambda > 0$	spiral source (unstab)		
$Re\,\lambda < 0$	spiral sink (stable)		

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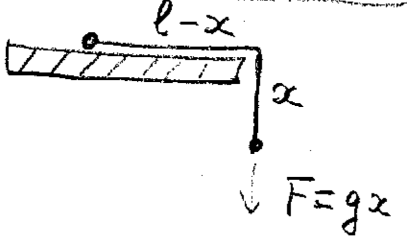
Applications of LDS

1. Sliding chain



DE: $\begin{cases} \ddot{x} = \frac{g}{\ell} x \\ x(0) = b, \ \dot{x}(0) = 0 \end{cases} \to X = \binom{x}{v}$

DS: $\begin{cases} \dot{X} = \begin{bmatrix} 0 & 1 \\ g/\ell & 0 \end{bmatrix} X \\ X(0) = \binom{b}{0} \end{cases}$

Eigenvalue: $\alpha = \sqrt{g/\ell} \ \Big| \ -\sqrt{g/\ell}$; $\binom{1}{\alpha} \ \Big| \ \binom{1}{-\alpha}$ $\Rightarrow \boxed{X = c_1 e^{\alpha t}\binom{1}{\alpha} + c_2 e^{-\alpha t}\binom{1}{-\alpha}}$

IVP: $c_1 = c_2 = b/2 \Rightarrow X = \frac{b}{2}\left[e^{\alpha t}\binom{1}{\alpha} + e^{-\alpha t}\binom{1}{-\alpha}\right]$

DE sol: $x(t) = \frac{b}{2}(e^{\alpha t} + e^{-\alpha t}) = b \cosh \alpha t$

Sliding time: $x(t) = \ell \Rightarrow \boxed{t_{sl} = \frac{1}{\alpha}\cosh^{-1}(\ell/b) = \frac{1}{\alpha}\ln \frac{\ell + \sqrt{\ell^2 - b^2}}{b}}$

2. **Linear model of competition-cooperation:** show that competing/cooperating species $\{x, y\}$ with the natural growth rates 2, 3, and interaction coefficients $\{b, c\}$ obey a differential system with matrix $A = \begin{bmatrix} 2 & \pm b \\ \pm c & 3 \end{bmatrix}$.

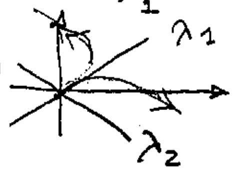
$\begin{bmatrix} 2 & \pm a \\ \pm b & 3 \end{bmatrix} \to \lambda^2 - 5\lambda + [6 - ab]$

$\lambda = \frac{5 \pm \sqrt{1 + 4ab}}{2}$; $\boxed{\begin{matrix} \lambda_1 < 2 \\ \lambda_2 > 3 \end{matrix}}$

$\begin{bmatrix} 2 & a \\ b & 3 \end{bmatrix}$: $X_{1,2} = \binom{-a}{\boxed{2-\lambda}}$;

– cooperation

$\begin{bmatrix} 2 & -a \\ -b & 3 \end{bmatrix}$: $X_{1,2} = \binom{a}{2 - \lambda_{1,2}}$



– competition

Bifurcations of LDS

Space of 2×2 matrices is large 4D $\left\{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\}$. But only 2 parameters:

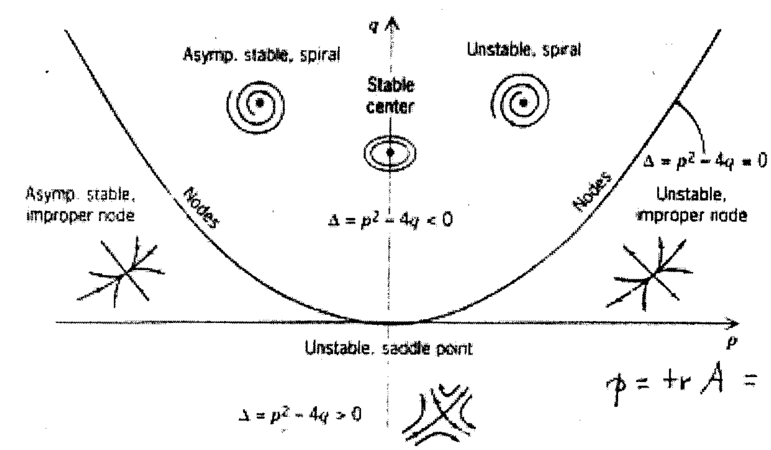
$$\left.\begin{array}{l} p = \operatorname{tr} A = a + d \\ q = \det A = ad - bc \end{array}\right\}$$ determine different dynamic patterns (phase-plane) equilibria type

as

$$\lambda_{1,2} = \frac{\operatorname{tr}}{2} \pm \sqrt{\frac{\operatorname{tr}^2}{4} - \det}$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$;

$q = \det A = ad - bc$



$p = \operatorname{tr} A = a + d$

FIGURE 9.1.9 Stability diagram.

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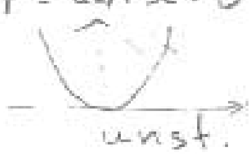
Activator - inhibitor systems

In many biological systems different components affect each other growth/decay

Example: Glucose in blood stimulates production of Insulin; Insulin removes Glucose

Marking
- $\longrightarrow\blacksquare$ - activation/stimulation
- $\longrightarrow\square$ - inhibition/removal
- $\circlearrowright$ - growth
- $\dashrightarrow$ - decay

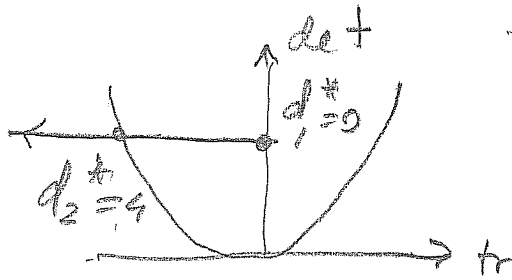
		Glucose - Insulin
$a \circlearrowright x \underset{b}{\overset{c}{\rightleftarrows}} y \overset{d}{\dashrightarrow}$	$\circlearrowright x \rightleftarrows y \circlearrowright$	$S \rightarrow x \rightleftarrows y \dashrightarrow$
$\begin{cases}\dot{x} = ax + by \\ \dot{y} = cx - dy\end{cases}$ $\begin{bmatrix} + & + \\ + & - \end{bmatrix}$	$\begin{cases} ax + by \\ -cx + dy \end{cases}$	$\begin{cases}\dot{x} = S - by \\ \dot{y} = cx - dy\end{cases}$
$A = \begin{bmatrix} a & b \\ c & -d \end{bmatrix}$	$\begin{bmatrix} a & b \\ -c & d \end{bmatrix}$	$\begin{bmatrix} 0 & -b \\ c & -d \end{bmatrix} \cdot Y + \begin{pmatrix} S \\ 0 \end{pmatrix}$
$\begin{cases} tr = a - d \gtrless 0 \\ det = -(ad + bc) < 0 \end{cases}$ saddle 	$tr = a + d > 0$ $det = ad + bc > 0$  unst.	$tr = -d < 0$ $det = bc > 0$  stable

Examples of LDS bifurcation

1. Damped oscillator:

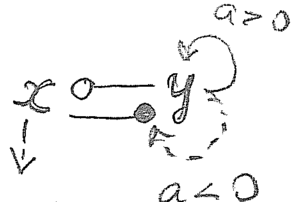
$$A = \begin{bmatrix} 0 & 1 \\ -4 & -d \end{bmatrix};\ d \geq 0$$

$$\begin{cases} tr = -d \\ det = 4 \end{cases}$$

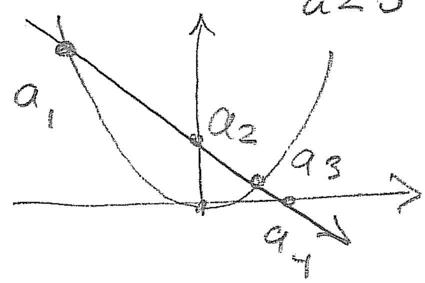


$d = 0$	$0 < d < 4$	$4 < d$
cent.	sp. sink	sink

2. $A = \begin{bmatrix} -1 & -1 \\ 2 & a \end{bmatrix}$;



$$\begin{cases} tr = a-1 \\ det = 2-a \end{cases}$$



Bif. values a^*

$$\boxed{\begin{aligned} a_1 &= -1 - 2\sqrt{2} \\ a_2 &= 1 \\ a_3 &= -1 + 2\sqrt{2} \\ a_4 &= 2 \end{aligned}}$$

3. $A = \begin{bmatrix} a & -1 \\ 2 & 0 \end{bmatrix}$; $tr = a$, $det = 2$