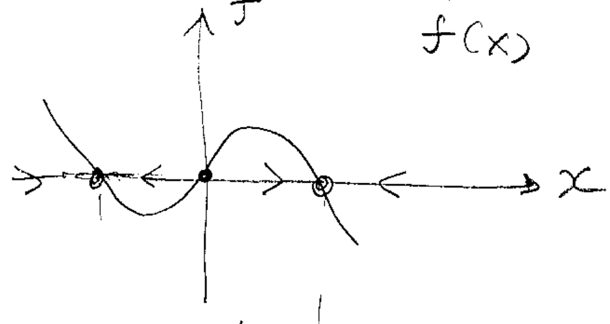
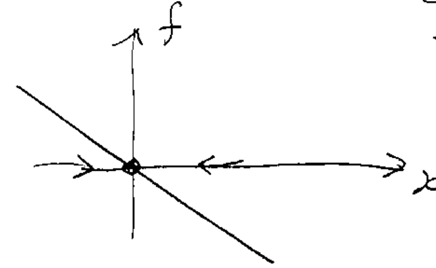
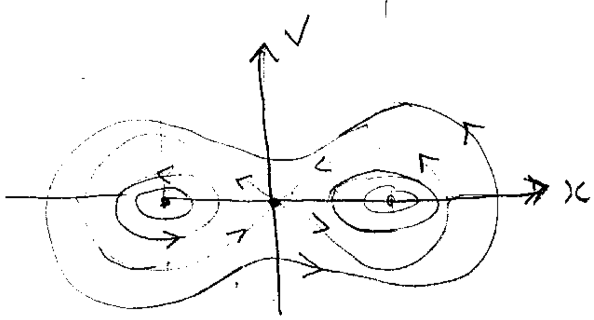
M224:

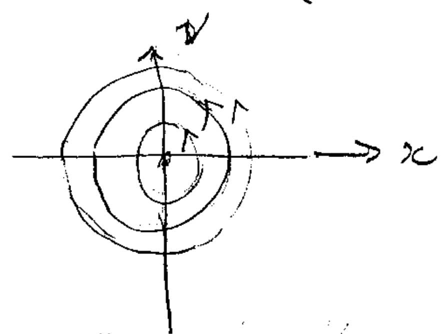
Oscillators

Duffing: $\begin{cases} \dot{x} = v \\ \dot{v} = \underbrace{ax - bx^3}_{f(x)} \end{cases}$; **Lin. osc:** $\begin{cases} \dot{x} = v \\ \dot{v} = -\underbrace{kx}_{f(x)} \end{cases}$



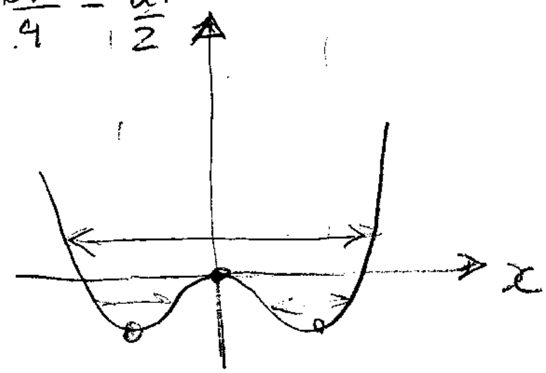






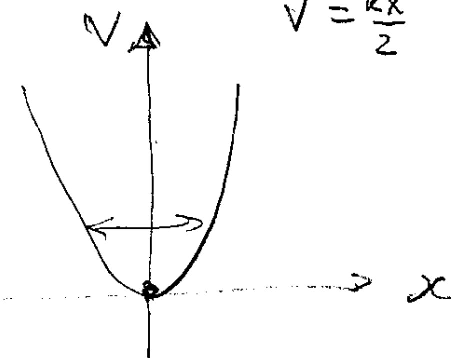
Potential V: $\boxed{f(x) = -V'(x)}$

$V = \frac{bx^4}{4} - \frac{ax^2}{2}$



double well

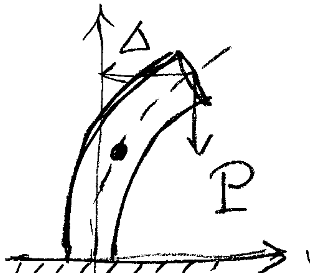
$V = \frac{kx^2}{2}$



single well

D. GURARIE

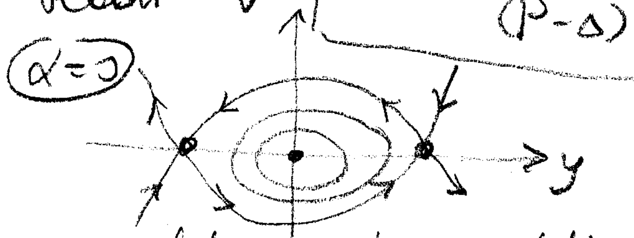
Swaying skyscraper (P-delta) and spring-pendulum

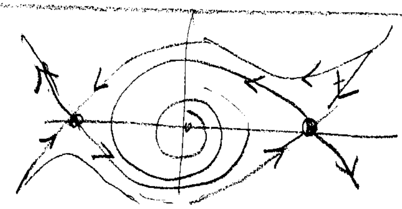


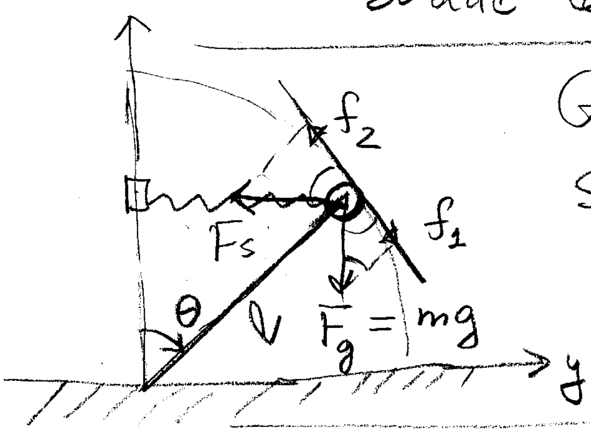
Buckled beam

⇒ Approximate eq-n:

$$\boxed{\ddot{y} = \underbrace{-ay + by^3}_{(P-\Delta)\text{ force}} - \underbrace{\alpha \dot{y}}_{\text{damping}}}$$







Gravity: $F_g = mg$

Spring: $F_s = -ky$

Projections: $f_1 = F_g \sin\theta = mg \sin\theta$

$f_2 = F_s \cos\theta = kl \sin \cos$

(1) $$\boxed{ml\ddot{\theta} = (-kl\cos\theta + mg)\sin\theta - \alpha\dot{\theta}}$$

Cubic approximation: $\cos\theta = 1 - \frac{\theta^2}{2} + \ldots$; $\sin\theta = \theta - \frac{\theta^3}{6} + \ldots$

(2) $$\boxed{\ddot{\theta} = \left[\left(\frac{g}{l} - \frac{k}{m}\right) + \frac{k}{2m}\theta^2\right]\theta}$$

Problem 2: (i) Derive cubic approximation (2) of (1)

(ii) Find equilibria, sketch phase plane and solution curves in 2 cases: $\frac{k}{m} > g/l$; $\frac{k}{m} < g/l$

(iii) Discuss stability of equilibria in both cases.

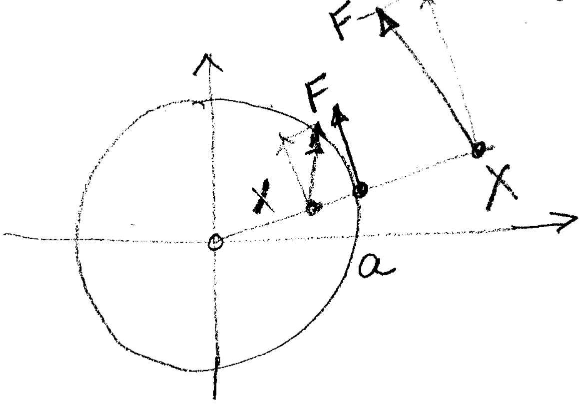
D. Gurarie

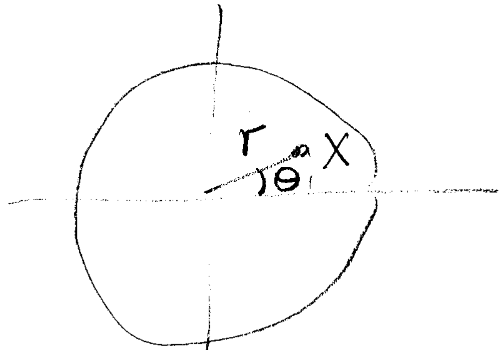
Limit cycle.

1) $\begin{cases} \dot{x} = -y + (a^2 - x^2 - y^2)x \\ \dot{y} = x + (a^2 - x^2 - y^2)y \end{cases}$

V. field

$F(X) = X^{\perp} + (a^2 - r^2)X$





Analytic solution in polar coord.

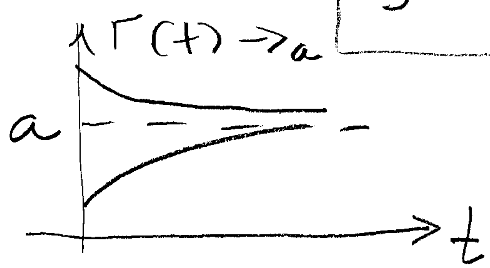
Polar change of variables: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

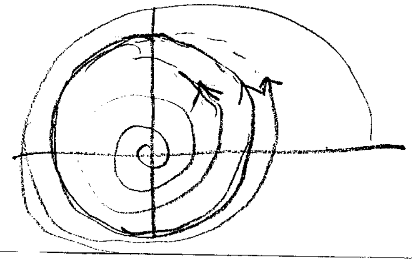
$$\Rightarrow \begin{cases} \dot{r} = r(a^2 - r^2) \\ \dot{\theta} = 1 \end{cases}$$ ⟷ decoupled

(i) $\displaystyle\int_{r_0}^{r} \frac{dr}{r(a^2 - r^2)} = t \Rightarrow \boxed{r(t) = a\,\frac{r_0 e^{a^2 t}}{\sqrt{(a^2 - r_0^2) + r_0^2 e^{2a^2 t}}}}$

(ii) $\boxed{\theta = t + \theta_0}$

$$\Rightarrow \boxed{\begin{aligned} x(t) &= r(t)\cos(t + \theta_0) \\ y(t) &= r(t)\sin(t + \theta_0) \end{aligned}}$$





Limit cycle
$r = a$
+ spirals