DS models

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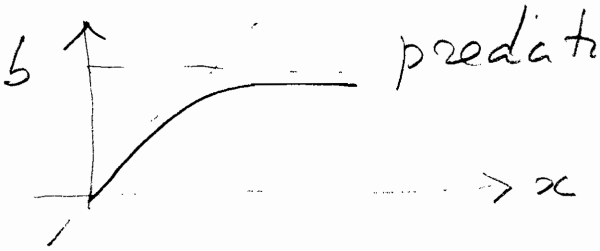
I. Population Biology

Pred.-prey

1) $\begin{cases} \dot{x} = ax - bxy - h \\ \dot{y} = cxy - dy \end{cases}$ Volterra–Lotka w. source/harvest

2) $\begin{cases} \dot{x} = a(1 - x/N)x - bxy \\ \dot{y} = cxy - dy \end{cases}$ Logistic prey

3) $\begin{cases} \dot{x} = a(1 - x/N)x - b\dfrac{x}{x+x_0}\,y \\ \dot{y} = c\dfrac{x}{x+x_0}\,y - dy \end{cases}$ Predation w. satiation



predation f-n

Competion/cooperation

$$\begin{cases} \dot{x} = a_1(1 - x/N_1)x \mp b_1 xy \\ \dot{y} = a_2(1 - x/N_2)y \mp b_2 xy \end{cases}$$

Food chain: $\begin{cases} \dot{x} = ax - bxy \\ \dot{y} = cxy - dyz \\ \dot{z} = eyz - fz \end{cases}$

"y" predates on "x", "z" predates on "y"

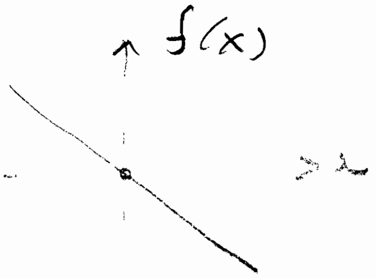
Mechanical systems (oscillators)

DE $\boxed{m\ddot{x} = \underset{\substack{\uparrow\\ \text{potential}\\ \text{force}}}{f(x)} - \underset{\substack{\uparrow\\ \text{friction}}}{d(\dot{x})} + \underset{\substack{\uparrow\\ \text{External}}}{b(t)}} \iff$ DS $\begin{cases} \dot{x} = v \\ \dot{v} = \frac{1}{m}\underbrace{F(x, v, t)}_{\text{combined force}} \end{cases}$

Examples:

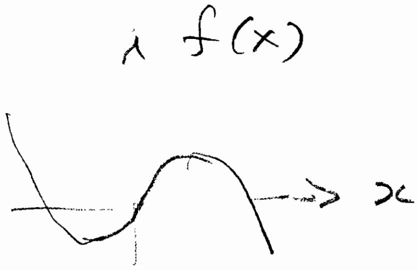
1) Linear oscill: $\boxed{f = -kx}$

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{k}{m}x - \frac{d}{m}v + b(t) \end{cases}$$

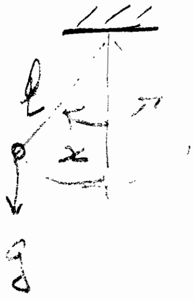


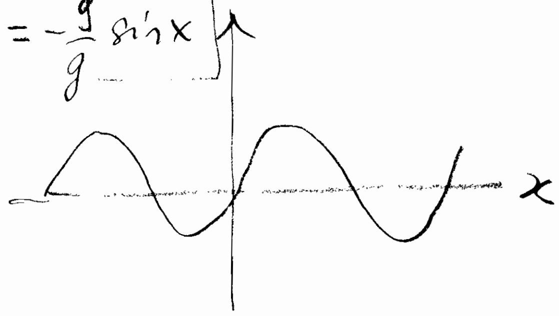
2) Duffing oscill: $\boxed{f = ax - bx^3}$

$$\begin{cases} \dot{x} = v \\ \dot{v} = ax - bx^3 - dv + \ldots \end{cases}$$



3) Pendulum: $\boxed{f = -\frac{g}{g}\sin x}$





$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{g}{\ell}\sin x - dv + \end{cases}$$

4) Non-lin. x-dependent friction (non-mech.)

$$\begin{cases} \dot{x} = v \\ \dot{v} = -kx + (1-x^2)v \end{cases}$$

van der Pole

Chemical reactions

(1) $A + B \underset{k_2}{\overset{k_1}{\rightleftarrows}} C$ - reactants

$x \quad y \quad z$ - concentrations

k_1, k_2 - reaction rates (binding, dissoc.)

$$\text{DS}\begin{cases} \dot{x} = -k_1 xy + k_2 z \\ \dot{y} = -k_1 xy + k_2 z \\ \dot{z} = k_1 xy - k_2 z \end{cases}$$

Mass conserv.

$x + z = N$ - const

$y + z = M$ - const

$$\Longrightarrow \boxed{\dot{z} = k_1 (N - z)(M - z) - k_2 z} \quad \text{DE}$$

Problem: Show DE (& DS) has stable equilibrium z^*. Find it

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(2) Ozone + NO_x photochemistry

$O + O_2 \to O_3$

$NO + O_3 \to NO_2 + O_2$

$NO_2 \xrightarrow{h\nu} NO + O$

Reduced scheme:

$O \xrightarrow{k_0} O_3$

$NO + O_3 \xrightarrow{k_1} NO_2$

$NO_2 \xrightarrow{k_2} NO + O$

O	u	spp/ concent
O_3	x	
NO	y	
NO_2	z	

$$\begin{cases} \dot{u} = -k_0 u + k_2 z \\ \dot{x} = k_0 u - k_1 y x \\ \dot{y} = -k_1 y x + k_2 z \\ \dot{z} = k_1 xy - k_2 z \end{cases}$$

$\boxed{k_0 u \approx k_2 z}$ as $k_0 \gg k_{1,2}$

$$\Rightarrow \begin{cases} \dot{x} = k_2 z - k_1 y x \\ \dot{y} = -k_1 xy + k_2 z \\ \dot{z} = k_1 xy - k_2 z \end{cases} \Leftrightarrow x + y \underset{k_2}{\overset{k_1}{\rightleftarrows}} z$$

Like system (1)

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Activator - inhibitor systems

In many biological systems different components affect each other growth/decay

Example: Glucose in blood stimulates production of Insulin; Insulin removes Glucose

Marking:
- —■ - activation/stimulation
- —□ - inhibition/removal
- ↻ - growth
- - - → - decay

$a \circlearrowleft x \underset{b}{\overset{c}{\rightleftarrows}} y \overset{d}{\dashrightarrow}$

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx - dy \end{cases} \quad \begin{bmatrix} + & + \\ + & - \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & -d \end{bmatrix}$$

$$\begin{cases} tr = a - d \gtrless 0 \\ det = -(ad + bc) < 0 \end{cases}$$

saddle 

$\circlearrowleft x \rightleftarrows y \circlearrowright$

$$\begin{cases} ax + by \\ -cx + dy \end{cases}$$

$$\begin{bmatrix} a & b \\ -c & d \end{bmatrix}$$

$tr = a + d > 0$

$det = ad + bc > 0$

unst.

Glucose - Insulin

$S \rightarrow x \rightleftarrows y \dashrightarrow$

$$\begin{cases} \dot{x} = S - by \\ \dot{y} = cx - dy \end{cases}$$

$$\begin{bmatrix} 0 & -b \\ c & -d \end{bmatrix} \cdot Y + \begin{pmatrix} S \\ 0 \end{pmatrix}$$

$tr = -d < 0$

$det = bc > 0$

stable

Epidemics: SIR - model

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$$\begin{tikzcd} S \arrow[r, "\beta"] & I \arrow[r, "\mu"] & R \arrow[ll, bend right, "\delta"'] \end{tikzcd}$$

susceptible infected recovered (immune)

Total pop. density: $S + I + R = N$ - const

Transmission / recovery DS

$$(1)\begin{cases} \dot{S} = -\beta S I + \delta R \\ \dot{I} = \beta S I - \mu I \\ \dot{R} = \mu I - \delta R \end{cases}$$

$\beta = \dfrac{\text{transm. rate}}{\text{per } I \text{ per } S}$

μ = recovery rate ($\frac{1}{\mu}$ = duration)

δ = loss of immunity rate

Tot. pop. density N - const reduces (1) to 2D system: $R = N - S - I$

$$(2)\begin{cases} \dot{S} = -\beta S I + \delta (N - S - I) \\ \dot{I} = \beta S I - \mu I \end{cases}$$

Plots below illustrate SIR for typical flu parameters: $\mu = .15$ $\left(\frac{1}{7 \text{ days}}\right)$

$\delta = .004$ (loss of immunity ≈ 250 days)

pop. density $N = 1$. Vary β

Problem: 1) Find equilibria, plot solution curves and phase-plane for $\beta = 3; 1; .1$

2) Estimate recurrence period (based on plots) in all 3 - cases.

3) Extra: study the effect of delta (= 250,90,30 days) on equilibria and reccurence.