Math. 244 supplement:
Variation of parameters, fundamental solution and convolution integrals

David Gurarie

1 Variation of parameters.

Variation of parameter method gives a particular solution of the inhomogeneous second order DE problem

$$L[y] = y'' + ay' + by = f$$

in terms of a fundamental pair $\{y_1; y_2\}$ of the homogeneous problem $L[y] = 0$

$$y_p(t) = -y_1(t)\int_{t_0}^{t}\frac{y_2(s)}{W(s)}f(s)\,ds + y_2(t)\int_{t_0}^{t}\frac{y_1(s)}{W(s)}f(s)\,ds \tag{1}$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ the Wronskian of two solutions. We combine two integrals (1) together and write it as

$$y_p(t) = \int_{t_0}^{t}\frac{y_2(t)\,y_1(s) - y_1(t)\,y_2(s)}{W(s)}f(s)\,ds = \int_{t_0}^{t}K(t,s)\,f(s)\,ds \tag{2}$$

Function

$$K(t,s) = \frac{y_2(t)\,y_1(s) - y_1(t)\,y_2(s)}{W(s)} \tag{3}$$

is called the *fundamental (impulse) solution* of the initial value problem

$$L[y] = f;\; y(0) = y'(0) = 0$$

It has two basic properties:

- function $K(t,s)$ solves a differential equation in variable t with δ-source : $L[K(t,...)] = \delta(t-s)$
- Any source f in the r.h.s. can be represented by integral (2) - a linear superposition principle.

2 Constant coefficient differential equations.

For constant coefficient differential operator L the fundamental pair is made of 2 exponentials $\{e^{\lambda_1 t}; e^{\lambda_2 t}\}$. So (3) becomes

$$K(t,s) = K(t-s) = \frac{e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)}}{\lambda_1 - \lambda_2} \tag{4}$$

In special cases we get

$$K(t) = \begin{cases} \frac{\sin\beta t}{\beta} & \text{for complex roots: } \pm i\beta \\ e^{\alpha t}\frac{\sin\beta t}{\beta} & \text{for complex roots: } \alpha \pm i\beta \\ te^{\alpha t} & \text{for repeated roote: } \lambda_1 = \lambda_2 = \alpha \end{cases}$$

For constant coefficient differential equations fundamental solution K is a function of the difference $t-s$ only, so (2) becomes the so called *convolution integral*

$$Y(t)=\int_0^t K(t-s)\,f(s)\,ds=K*f \tag{5}$$

Notice that function $K(t)$ itself solves a homogeneous equation $L[K]=0$ with initial values:

$$K(0)=0;\ K'(0)=1$$

and the convolution integral solution (5) has zero initial values: $Y(0)=Y'(0)=0$. Hence the general IVP

$$\begin{aligned} L[y]&=f\\ y(0)&=a\\ y'(0)&=b \end{aligned} \tag{6}$$

has solution

$$y=\underbrace{K*f}_{Y_p}+\underbrace{c_1y_1+c_2y_2}_{y_h}$$

where the contribution of the r.h.s. (forcing term) is separated from the initial data, i.e. Y_p accounts for the r.h.s. and zero initial data, while homogeneous part y_h gives the contribution of the initial data and zero r.h.s.

The same results are obtained by the Laplace transform method with δ-source. Indeed, $L[y]=\delta(t)$ with zero initial data gives fundamental solution

$$\begin{aligned} y(t)&=\mathcal{L}^{-1}\left[\frac{1}{p(s)}\right]=\mathcal{L}^{-1}\left[\frac{1}{\lambda_1-\lambda_2}\left(\frac{1}{s-\lambda_1}-\frac{1}{s-\lambda_2}\right)\right]\\ &=\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}=K(t) \end{aligned}$$

while arbitrary source $L[y]=f$ has solution

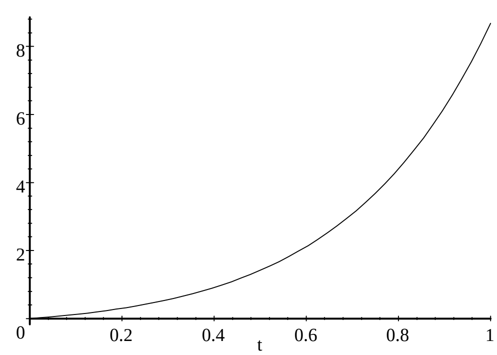
$$y=\mathcal{L}^{-1}\left[\frac{1}{p(s)}F(s)\right]=K*f$$

3 Examples

We compute and plot fundamental solution K and convolution integrals in several cases.

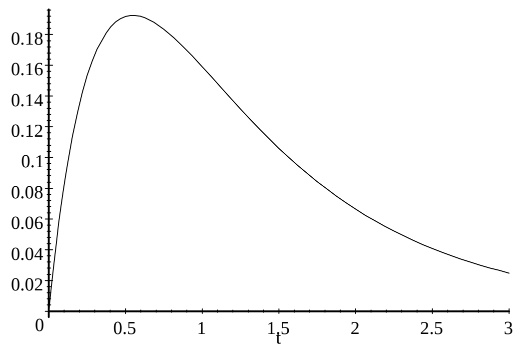
Positive characteristic roots:

$$y''-4y'+3y=f\Rightarrow K(t)=\frac{e^{3t}-e^{t}}{2}$$

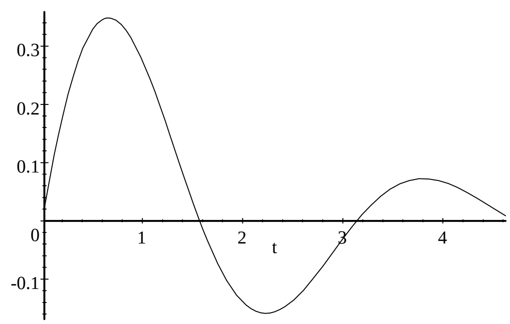


Negative roots (over-damped oscillator):

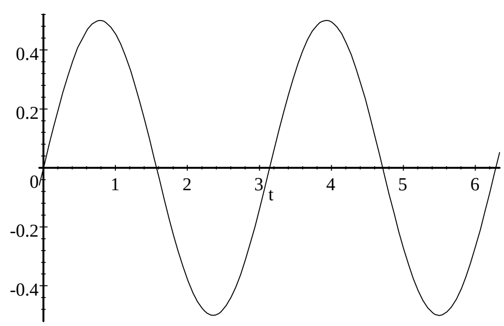
$$y'' + 4y' + 3y = f \Rightarrow K(t) = \frac{e^{-t} - e^{-3t}}{2}$$



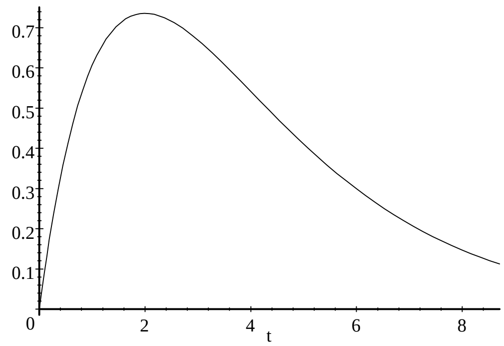
Complex roots (under-damped oscillator): $y'' + y' + 4.25y = f \Rightarrow K = e^{-.5t} \frac{\sin 2t}{2}$



Imaginary roots (undamped oscillator): $y'' + \omega^2 y = f \Rightarrow K = \frac{\sin \omega t}{\omega}$



Repeated roots: $y'' + y' + .25y = f \Rightarrow K = te^{-.5t}$



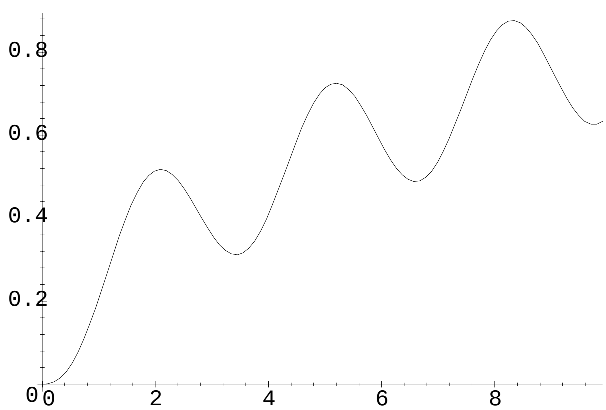
All satisfy initial conditions: $K(0) = 0$, slop $K'(0) = 1$

We also compute the convolution integral for the $\sqrt{t}$-forced oscillator with zero initial data: $y'' + 4y = \sqrt{t}$

$$y(t) = \int_0^t \frac{\sin 2(t-s)}{2} \sqrt{s} ds$$

$$y = \frac{\sqrt{t}}{4} - \frac{\sqrt{\pi}}{4} \operatorname{FresnelC}\left(\frac{2\sqrt{t}}{\sqrt{\pi}}\right) \cos^2 t + \frac{\sqrt{\pi}}{8} \operatorname{FresnelC}\left(\frac{2\sqrt{t}}{\sqrt{\pi}}\right) - \frac{\sqrt{\pi}}{4} \operatorname{FresnelS}\left(\frac{2\sqrt{t}}{\sqrt{\pi}}\right) \sin t \cos t$$

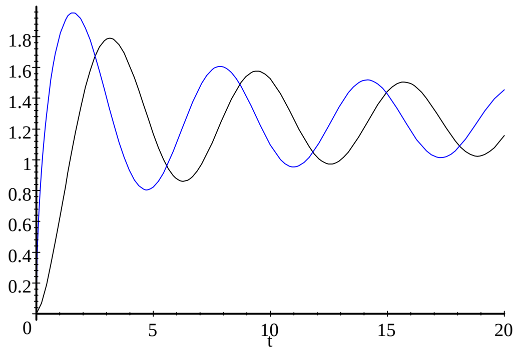
-a scary looking special function plotted below



The basic Fresnel integrals are defined as

$$\int_0^t \frac{\sin s}{\sqrt{s}} ds = \sqrt{2\pi} \operatorname{FresnelS}\left(\sqrt{\frac{2t}{\pi}}\right)$$

$$\int_0^t \frac{\cos s}{\sqrt{s}} ds = \sqrt{2\pi} \operatorname{FresnelC}\left(\sqrt{\frac{2t}{\pi}}\right)$$


1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
5
10
15
20
t