M224
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Damped oscillators (Ch.3)

$$m\ddot{x} + d\dot{x} + kx = 0 \Rightarrow \boxed{\phi = m\lambda^2 + d\lambda + k = 0}$$

d – damping; ϕ – Charact.

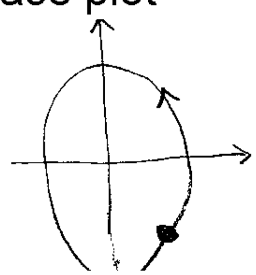
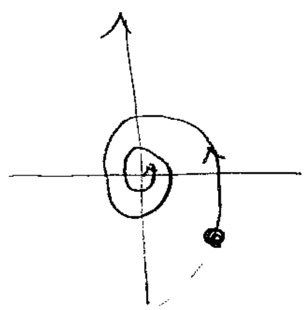
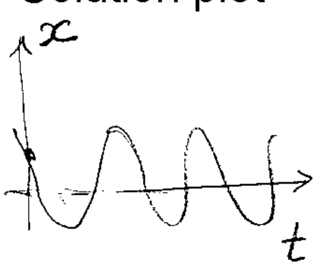
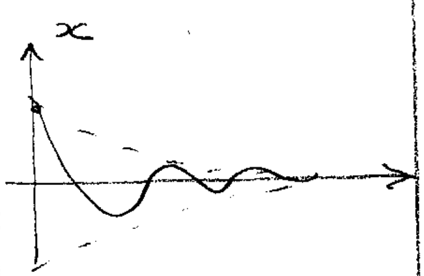
Roots: $\lambda = -\frac{d}{2m} \pm \sqrt{\left(\frac{d}{2m}\right)^2 - \frac{k}{m}} \Rightarrow \boxed{d_{cr} = 2\sqrt{km}}$ - critical

Below d_{cr} - complex, above d_{cr} - real

Damping	λ		GS $x(t)$
$d=0$	$\pm i\omega$	oscill. freq. $\omega = \sqrt{k/m}$	$c_1 \cos\omega t + c_2 \sin\omega t$
$0<d<d_{cr}$	$-\alpha \pm i\beta$	$\alpha = \frac{d}{2m}$ - damping rate; $\beta = \sqrt{k/m - (d/2m)^2} < \omega$ - freq	$e^{-\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$
$d=d_{cr}$	$-\alpha$	$\alpha = \frac{d}{2m}$ - crit. damped	$e^{-\alpha t}(c_1 + c_2 t)$
$d>d_{cr}$	$\lambda_{1,2} = -\alpha \pm \beta$	overdamped $\beta = \sqrt{\left(\frac{d}{2m}\right)^2 - k/m}$	$c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t} = e^{-\alpha t}(c_1 \cosh\beta t + c_2 \sinh\beta t)$

IVP solution $\begin{cases} m\ddot{x} + d\dot{x} + kx = 0 \\ x(0) = x_0; \quad \dot{x}(0) = v_0 \end{cases} \Rightarrow c_1 x_1(t) + c_2 x_2(t)$

$$x(t) = \begin{cases} x_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t & \text{— undamped} \\ e^{-\alpha t}\left[x_0 \cos\beta t + \left(\frac{v_0 + \alpha x_0}{\beta}\right)\sin\beta t\right] & \text{— underdamped} \\ e^{-\alpha t}\left[x_0 + (v_0 + \alpha x_0)\, t\right] & \text{— crit. damped} \\ e^{-\alpha t}\left[x_0 \operatorname{ch}\beta t + \left(\frac{v_0 + \alpha x_0}{\beta}\right)\operatorname{sh}\beta t\right] & \text{— overdamped} \end{cases}$$

DE $\ddot{x} + 4x = 0$	$\ddot{x} + 2\dot{x} + 4x = 0$	$\ddot{x} + 4\dot{x} + 4x = 0$	$\ddot{x} + 6\dot{x} + 4x = 0$
Char. polyn. p= $\lambda^2 + 4 = 0$ $\lambda = \pm 2i$	$\lambda^2 + 2\lambda + 4 = 0$ $\lambda = -1 \pm \sqrt{3}i$	$\lambda^2 + 4\lambda + 4 = 0$ $\lambda = -2$	$\lambda^2 + 6\lambda + 4 = 0$ $\lambda_{1,2} = -1, -4$
Gen. sol. $c_1 \cos 2t + c_2 \sin 2t$	$e^{-t}(c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t)$	$e^{-2t}(c_1 + c_2 t)$	$c_1 e^{-t} + c_2 e^{-4t}$ $= e^{-3t}[c_1 \operatorname{ch} t + c_2 \operatorname{sh} t]$
Coefficients $\begin{cases} c_1 = 1 \\ 2c_2 = -1 \end{cases}$	$\begin{cases} c_1 = 1 \\ -c_1 + \sqrt{3}c_2 = -1 \end{cases}$		
IVP solution $\cos 2t - \frac{\sin 2t}{2}$			
Phase plot 			
Solution plot 			

Problem: Complete the table (you can use computer plots).

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Characteristic polynomial for linear DEs: $L[y]=f$

Diff. operation $L=\begin{cases} aD+b \\ aD^2+bD+c \\ \dots \end{cases}$ has charact. polyn. $p(\lambda)=\begin{cases} a\lambda+b \\ a\lambda^2+b\lambda+c \\ \dots \end{cases}$

It allows to solve both homogeneous $L[y]=0$ and "forced" (inhomog.) problem: $L[y]=f(t)$

1) Homog. case: $L[y]=0 \Rightarrow \boxed{p(\lambda)=0} \Rightarrow \begin{cases} \lambda_1 \to y_1=e^{\lambda_1 t} \\ \lambda_2 \to y_2 = e^{\lambda_2 t} \\ \vdots \end{cases}$ charact. roots

General sol. $\boxed{y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots}$

2) Exponential forcing: $L[y]=f_0 e^{\mu t} \Rightarrow$

Partic. sol: $\boxed{y_p = \dfrac{f_0}{p(\mu)} e^{\mu t}}$ provided $\mu \neq \lambda_1, \lambda_2 \dots$ – non charact

Gen. sol: $y = \dfrac{f_0}{p(\mu)} e^{\mu t} + c_1 e^{\lambda_1 t} + \dots$ IVP: $\begin{cases} y(0) = \dfrac{f_0}{p(\mu)} + c_1 + \dots \\ y'(0) = \dfrac{\mu f_0}{p(\mu)} + \lambda_1 c_1 + \dots \end{cases}$

Examples:

1) $\begin{cases} y' = 2y + e^{\mu t} \\ y(0) = -1 \end{cases}$	$p=\lambda-2$ $\lambda_1 = 2$	$y = ce^{2t} + \frac{1}{\mu-2} e^{\mu t}$ $(\mu \neq 2)$	$c + \frac{1}{\mu-2} = -1$
2) $\begin{cases} y''+4y = 3e^{\mu t} \\ y(0)=1;\ y'(0) = -2 \end{cases}$	$p = \lambda^2+4$ $\lambda_{1,2} = \pm 2i$	$y = \frac{3}{\mu^2+4} e^{\mu t} + (c_1 \cos 2t + c_2 \sin 2t)$ $\mu \neq \pm 2i$	$\frac{3}{\mu^2+4} + c_1 = 1$ $\frac{3\mu}{\mu^2+4} + c_2 = -2$
3) $\begin{cases} y''+2y'+5y = 3e^{\mu t} \\ y(0)=1;\ y'(0) = -2 \end{cases}$	$\lambda_{1,2} = -1 \pm 2i$	$y = \frac{3e^{\mu t}}{\mu^2+2\mu+5} + e^{-t}(c_1 \cos 2t + c_2 \sin)$	$\begin{cases} \frac{3}{\mu^2+\dots} + c_1 = 1 \\ \frac{3\mu}{\mu^2+\dots} + \dots = -2 \end{cases}$

Periodic force

DE: $L[y] = \begin{cases} f_0 \cos\omega t \\ f_0 e^{\alpha t} \cos\omega t \end{cases} = \begin{cases} f_0 \, Re\left[e^{i\omega t}\right] \\ f_0 \, Re\left[e^{(\alpha+i\omega)t}\right] \end{cases}$

1° Solve DE $L[y] = e^{\mu t}$ with complex $\mu = \begin{cases} i\omega \\ \alpha + i\omega \end{cases}$

2° Take Re: $\boxed{y_p = f_0 \, Re\left[\dfrac{e^{\mu t}}{p(\mu)}\right]}$. Expand $p(\mu) = P + iQ$

$$\Rightarrow \quad y_p = f_0 e^{\alpha t}\left[\frac{P\cos\omega t + Q\sin\omega t}{P^2+Q^2}\right] = \underbrace{\frac{f_0}{\sqrt{P^2+Q^2}}}_{\text{amplit.}} \cos(\omega t - \underbrace{\phi}_{\text{ph.-shift}}) e^{\alpha t}$$

$$\boxed{y_p = e^{\alpha t} A\cos(\omega t - \phi)} \qquad A = \frac{f_0}{|p(\mu)|} = \frac{f_0}{\sqrt{P^2+Q^2}}; \quad \phi = \tan^{-1}\left(\frac{Q}{P}\right)$$

Examples:

$\underbrace{y' - 2y}_{\text{per. harvest}} = f_0 \cos\omega t$	$f_0 \, Re\left[\dfrac{e^{i\omega t}}{i\omega - 2}\right]$	$\dfrac{f_0}{\omega^2+4}\left[\underbrace{-2\cos\omega t + \omega\sin\omega t}\right] = \dfrac{f_0}{\sqrt{4+\omega^2}}\cos(\omega t - \phi)$; $\phi = \tan^{-1}\left(\frac{\omega}{2}\right)$
$y'' + 4y = f_0 \cos\omega t$	$f_0 \, Re\left[\dfrac{e^{i\omega t}}{4-\omega^2}\right]$	$\dfrac{f_0}{4-\omega^2}\cos\omega t$
$y'' + 2y' + 5y = f_0 \cos\omega t$	$f_0 \, Re\left[\dfrac{e^{i\omega t}}{(5-\omega^2)+2i\omega}\right]$	$\dfrac{f_0 \cos(\omega t - \phi)}{\sqrt{(5-\omega^2)^2 + 4\omega^2}}$; $\phi = \tan^{-1}\left(\dfrac{2\omega}{5-\omega^2}\right)$

1) $\begin{cases} y' - 2y = .5\cos 3t \\ y(0) = 1 \end{cases} \Rightarrow y = \underbrace{\frac{.5}{13}\left[-2\cos 3t + 3\sin 3t\right]} + \underbrace{\left(1 + \frac{1}{13}\right)}_{c_1} e^{2t}$

$= \frac{.5}{\sqrt{13}}\cos(3t + \phi) + \frac{14}{13}e^{2t}, \quad \phi = \tan^{-1}\left(\frac{3}{2}\right)$

2) $\begin{cases} y'' + 4y = .5\cos 3t \\ y(0) = 1; \quad y'(0) = 0 \end{cases} \Rightarrow y = \underbrace{\boxed{\frac{.5}{(-5)}\cos 3t}}_{y_p} + \underbrace{1.1\cos 2t + 0\sin 2t}_{y_h}$

3) $\begin{cases} y'' + 2y' + 5y = .5\sin 3t \\ y(0) = 1; \quad y'(0) = 0 \end{cases} \Rightarrow$

$y = \underbrace{.5\left[\frac{(-4)\cos 3t - 6\sin 3t}{(-4)^2 + 6^2}\right]} + e^{-t}\left[\underset{?}{c_1}\cos 2t + \underset{?}{c_2}\sin 2t\right]$

$\frac{.5}{\sqrt{4^2 + 6^2}}\cos(3t - \phi) + e^{-t}A\cos(2t - \theta); \quad \theta = \tan^{-1}\left(\frac{c_2}{c_1}\right)$

$\phi = \tan^{-1}\left(\frac{6}{-4}\right)$