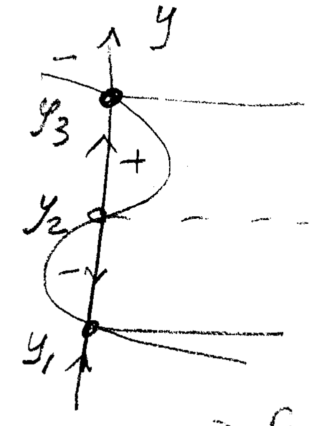
224 Phase-line; equilibria, linearization & stability for $y'=f(y)$ | D. GURARIE

1. Equilibria: (algebraic) $\boxed{f(y)=0 \Rightarrow \begin{cases} y_1 \\ y_2 \\ \vdots \end{cases}}$




2. Linearization: replace nonlinear (NLDE) $\boxed{y'=f(y)}$ with approximate linear (LDE): $\boxed{u'=mu}$ for $u(t)=y(t)-y_1$; and coeff. $\boxed{m=f'(y_1)}$ – slope

Equilibria types:

| $m$ | type |
|---|---|
| ⊖ | stable (sink) |
| ⊕ | unstabe (source) |
| 0 | ?? (semi-stable ...) |

Examples:

| $f(y)$ | Equil. | $f'(y)$ | $m$ | LDE | Approx. solutions |
|---|---|---|---|---|---|
| $ay(1-y/N)$ | 0 – source<br>N – sink | $1-\frac{2y}{N}$ | $a>0$<br>$-a<0$ | $u'=au$<br>$u'=-au$ | $u=Ce^{at}$ \| $y\approx Ce^{at}$<br>$u=Ce^{-at}$ \| $y\approx N+Ce^{-at}$ |
| $y^2(1-y)$ | 0 (double)<br>1 – sink | $2y-3y^2$ | 0?<br>$-1$ | $u'\overset{?}{=}0\cdot u$<br>$u'=-u$ | – ?? <br>$y\approx 1+Ce^{-t}$ |
| $y(1-\frac{y}{M})(1-\frac{y}{N})$ modified Logistic | 0 (sink)<br>M (source)<br>N (sink) | ... | $<0$<br>$>0$<br>$<0$ | ...<br>...<br>... | $y\approx 1+Ce^{-m_1 t}$<br>$y\approx$ ...<br>$y\approx$ ... |

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# Bifurcations

(*) Many DE involve other parameters: $\boxed{y' = f(y, a, \dots)}$.

Examples: 1) $y' = ay \pm b = f(y, a, b)$

2) $y' = ay(1 - y/N) \pm b = f(y, a, N, b)$

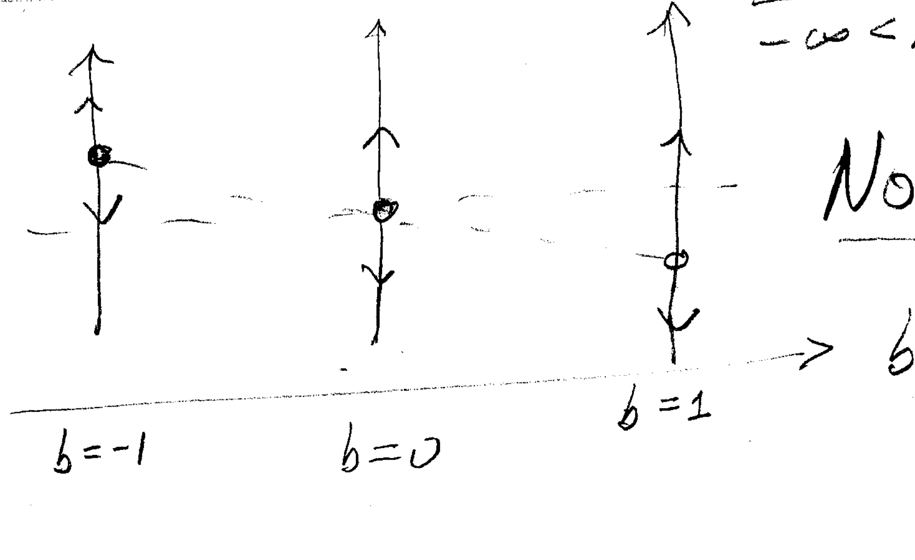
(*) Problem: how change(s) in parameters $a, \dots$ change qualitative solution pattern = Phase-line.

$\boxed{\text{"Change"} = \text{"bifurcation"(??)}}$

Examples:

1) LDE w. source/harvest: $\boxed{y' = y + b}$

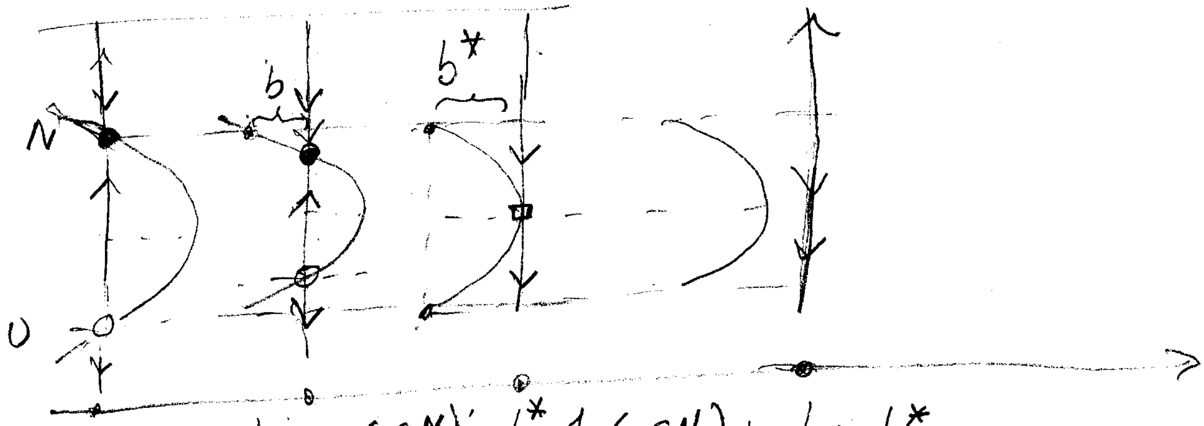
$-\infty < b < \infty$



No change!

## 2) Logistic population w. harvest

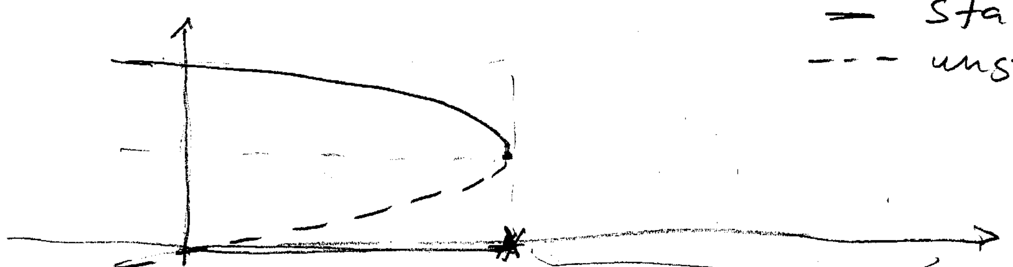
$$y' = ay(1 - y/N) - b \quad \text{or} \quad y(1-y) - b$$




$b = 0 \qquad b = .1 \left(< \frac{aN}{2}\right); \quad b^* = \frac{1}{4} \left(= \frac{aN}{2}\right); \quad b > b^*$

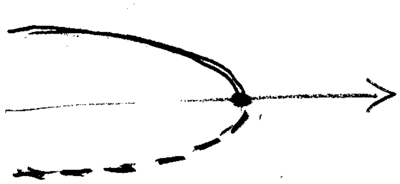
## Bifurcation diagram

— stable
--- unstable



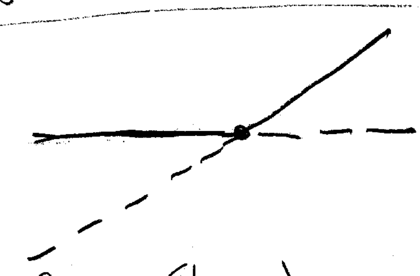

sustainable $\qquad b^* = \frac{aN}{2}$ bifurcation value $\qquad$ unsustainable

## Basic Bifurcation types (patterns) in 1D



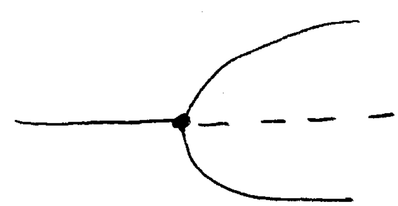
$f = y(1-y) - b$

saddle-node



$f = y(b-y)$

transcritical



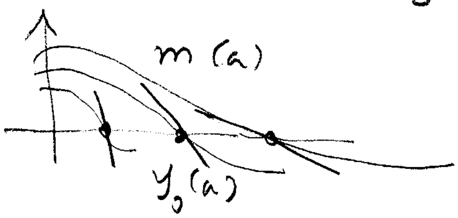
$f = y(b-y^2)$

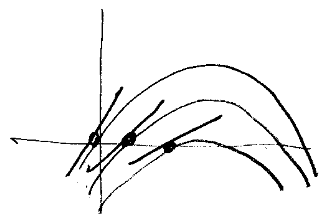
Pitchfork

224   # 1D bifurcations   D. GURARIE

DE: $\boxed{y' = f(y,a)}$;   Examples: $\left.\begin{array}{l} y' = f(y) - a \\ y' = f(y) - ay \end{array}\right\}$ harvest

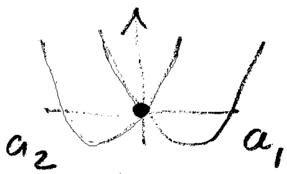
$$\begin{cases} y_0(a) & \leftarrow \text{equilibria} \; \ldots \\ m(a) = f_y(y_0, a) & \leftarrow \text{slope} \end{cases}$$



Bifurcation = change of type of equibria

Example:




$m(a_1) > 0$,   $m(a_2) < 0$

Condition for bifurc: $\boxed{m(a) = 0}$

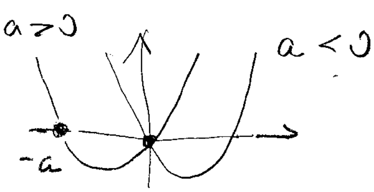
Bifurcation is an algebraic problem

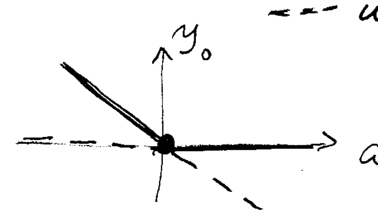
$$\begin{cases} f(y,a) = 0 & \leftarrow \text{equil.} \\ f_y(y,a) = 0 & \leftarrow \text{bifurc. } (m = 0) \end{cases}$$

— stable
--- unst.

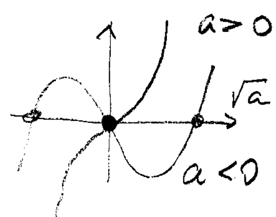
Types: 1) $ax + x^2$:

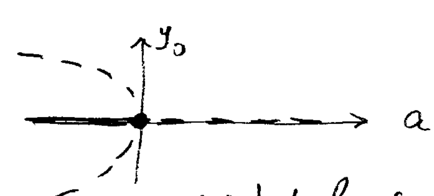
$f(x,a) = k\,x(1-x/N) - a\,x$




2) $ax + x^3$:



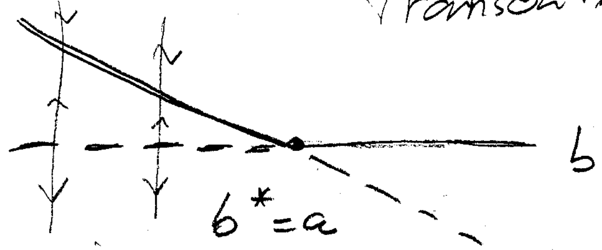

. . .

pitchfork

3)

# L3-3: Examples of bifurcations

Transcrit.



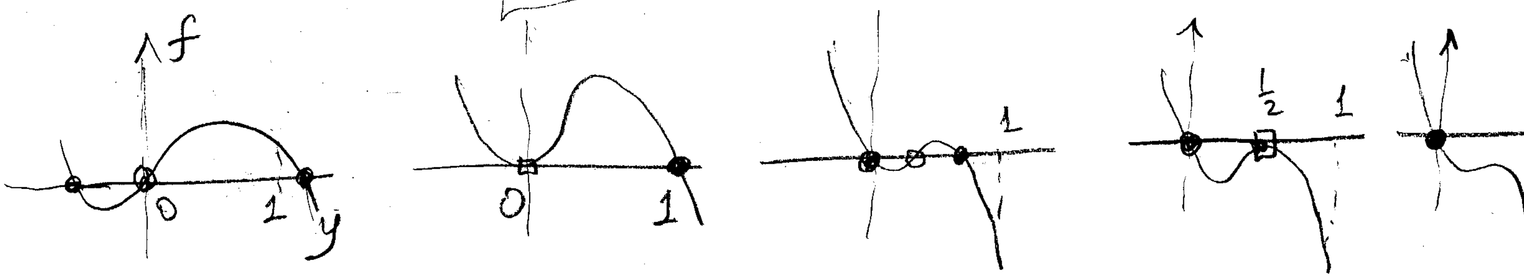

1) $y' = \underbrace{ay(1 - y/N)} - by$;

$$\begin{cases} f = ay\left[(1 - b/a) - y/N\right] = 0 \\ f_y = a\left[(1 - b/a) - \frac{2y}{N}\right] = 0 \end{cases} \ldots$$

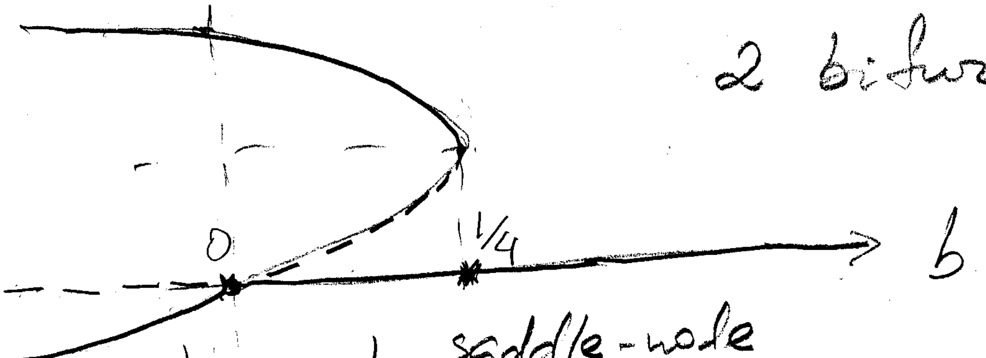
$b^* = a$
$y^* = 0$

2) $y' = \underbrace{y^2(1-y)} - by = y\left[y(1-y) - b\right]$; $-\infty < b < \infty$

$b<0$ $\quad$ $\boxed{b=0}$ $\quad$ $0<b<\frac{1}{4}$ $\quad$ $\boxed{b=\frac{1}{4}}$ $\quad$ $b>\frac{1}{4}$




2 bifurcations

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Analysis:

$$\begin{cases} f = y\left[y(1-y) - b\right] = 0 \\ f_y = 2y - 3y^2 - b = 0 \end{cases} \Rightarrow \ldots$$

| $y^*$ | $b^*$ |
|---|---|
| 0 | 0 |
| 1/2 | 1/4 |

D. GURARIE

# Models of epidemics

## 1. SI-model

$$\boxed{S} \underset{\mu}{\overset{\beta}{\rightleftarrows}} \boxed{I}$$

Population strata: suscept. infected

$$(SI) \quad \boxed{\begin{cases} \dfrac{dS}{dt} = -\beta S I + \mu I \\[2mm] \dfrac{dI}{dt} = \beta S I - \mu I \end{cases}}$$

$\beta$ – infection rate per unit $I$ unit time

$\mu$ – recovery rate $\left(\frac{1}{\mu} = \text{duration of illness}\right)$

$\beta S$ = number of $S$ infected by each $I$ per unit time

---

System (SI) conserves total population $S + I = N$ – const $\Rightarrow$ reduce to single DE

$$\text{(I)} \qquad \boxed{\dot{I} = \beta I (N - I) - \mu I} \quad \text{– logistic w. harvest}$$

$\beta$ = prevention $\qquad \mu$ = treatment

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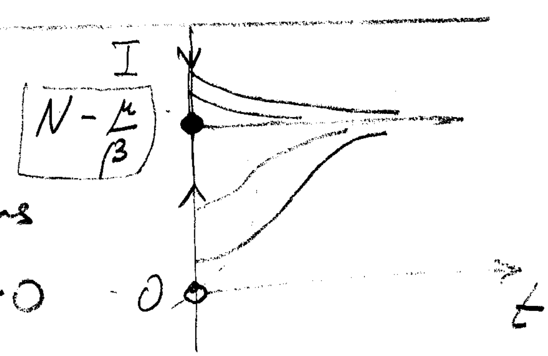
Equilibria:




(i) if $\frac{\mu}{\beta} < N \Rightarrow$ all solutions converge to $I^* = N - \mu/\beta > 0$ (endemic infection

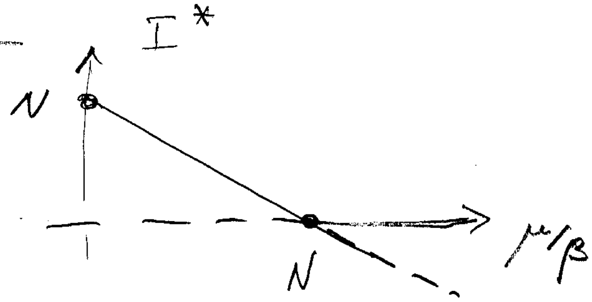
(ii) To eradicate infection need

$$\boxed{N \le \mu/\beta} \iff I^* \le 0.$$

2 parameters of control: $\mu$ – cure (increased $\mu$ shortens duration of illness)

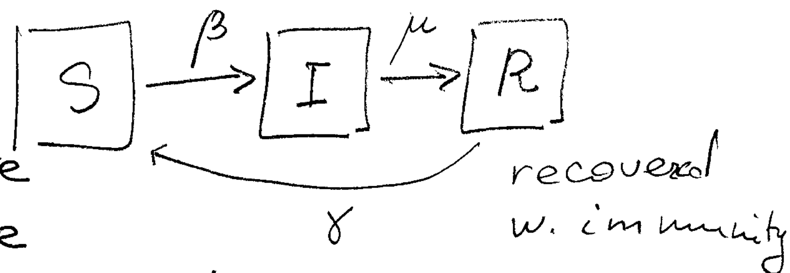
$\beta$ – transmission.

Bifurcation at $\frac{\mu}{\beta} = N$




Conclusion: Eradication is achieved by cure (increased $\mu$) or prevention (decreased $\beta$) so that $\frac{\mu}{\beta} \ge N$.

2. SIR-model




$\beta$ – transmission rate

$\mu$ – recovery rate

$\gamma$ – loss of immunit rate

Reduced via $S+I+R=N$

$$(SIR)\ \begin{cases} \dot S = -\beta S I + \gamma R \\ \dot I = \beta S I - \mu I \\ \dot R = \mu I - \gamma R \end{cases} \Rightarrow \begin{cases} \dot S = -\beta S I + \gamma(N - S - I) \\ \dot I = \beta S I - \mu I \end{cases}$$