D. GURARIE

Laplace transform method

1) **Goal:** * transform DE to AE (alg. eq-ns)

* Solve AE; tranform bak

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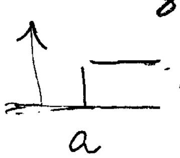
2) **Definition:**

$$\boxed{f(t)} \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \boxed{F(s)} = \int_0^\infty f(t)e^{-st}\,dt$$

Tables

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^n	$n!/s^{n+1}$
$e^{\alpha t}$	$1/s-\alpha$
$\cos\beta t$	$s/s^2+\beta^2$
$\sin\beta t$	$\beta/s^2+\beta^2$
$u_a(t)$	e^{-as}/s



Properties

f	F
Shift: $f(t-a)u_a$	$e^{-as}F(s)$
Multipl: $e^{at}f(t)$	$F(s-a)$
$t\,f(t)$	$-\frac{d}{ds}F(s)$
$f'=\frac{d}{dt}f$	$sF(s)-f(0)$
f''	$s^2F(s)-f'(0)-sf(0)$
...	...

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3) **DE solutions: partial fractions, inversion**

Take diff. op-tor $L = D+a$; D^2+aD+b; ...

1^0 IVP $\boxed{\begin{cases} L[y] = f(t) \\ y(0) = y_0 \\ \dots \end{cases}} \xrightarrow{\mathcal{L}}$ $\boxed{\underbrace{p(s)}_{\text{char. polyn.}}Y(s) = F(s) + \underbrace{\dots}_{\text{initial}}}$

DE $\qquad$ AE

2^0 $Y(s) = F(s)/p(s) + \dots/p(s)$

3^0 **Inversion:** $Y(t) \xrightarrow{\mathcal{L}^{-1}} y(t) = \mathcal{L}^{-1}\left[F(s)/p(s)\right] + \mathcal{L}\left[\frac{\dots}{p(s)}\right]$

Examples:

1) $\left\{\begin{array}{l} y' - 2y = 3e^{-t} \\ y(0) = 4 \end{array}\right]$ linear growth

Char. polyn. $p(\lambda) = \lambda - 2$

$$\Big\downarrow \mathcal{L}$$

1° $(s-2)Y(s) = \frac{3}{s+1} + 4$

2° $Y(s) = \frac{3}{(s+1)(s-2)} + \frac{4}{s-2}$

$\downarrow$ partial fractions

$-\frac{1}{s+1} + \frac{1}{s-2} + \frac{4}{s-2}$

3° $y(t) = \underbrace{\boxed{-e^{-t} + e^{2t}}}_{y_p} + \underbrace{\boxed{4e^{2t}}}_{y_h}$ (particular + homog.)

Comparison

* Undetermined: $\boxed{y(t) = \frac{3e^{-t}}{p(-1)} + c\,e^{2t}}$ (IV) $\Rightarrow \boxed{c=5}$

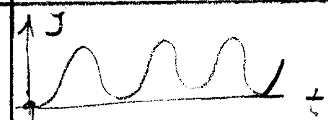
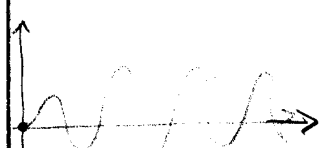
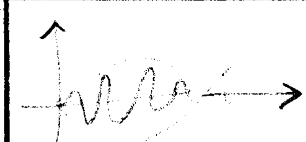
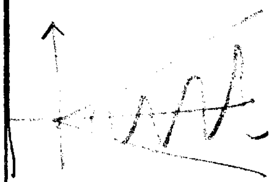
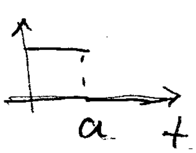
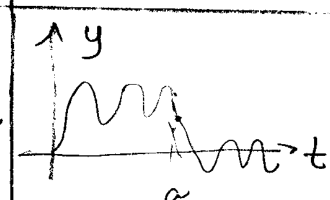
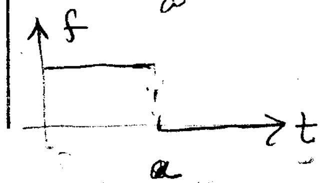
* multiplier: $y(t) = Ce^{2t} + \int_0^t e^{2(t-\tau)} 3e^{-\tau} d\tau = \dots = -e^{-t} + 4e^{2t}$

2) Undamped oscillator:

Steps: (1-2) (3)

$$\begin{cases} y'' + 4y = f(t) \\ y(0) = a \\ y'(0) = b \end{cases} \xrightarrow{\mathcal{L}} Y(s) = \frac{F(s)}{s^2+4} + \frac{b+as}{s^2+4} \xrightarrow{\mathcal{L}^{-1}} y_p(t) + \underbrace{a\cos 2t + \frac{b}{2}\sin 2t}_{y_h(t)}$$

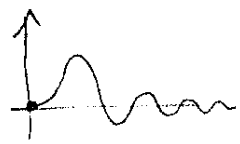
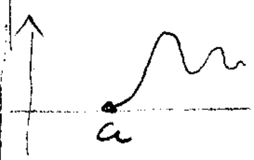
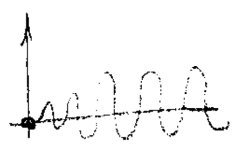
Particular 'Laplace solutions' for oscillator with several sources f(t)

f	$Y_p(s)$	$y_p(t)$	
1	$\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$	$\frac{1-\cos 2t}{4}$	
e^{-at}	$\frac{1}{(s+a)(s^2+4)} = \frac{1}{4+a^2}\left(\frac{1}{s+a} - \frac{s+a}{s^2+4}\right)$	$\frac{e^{-at} - \cos 2t + \frac{a}{2}\sin 2t}{4+a^2}$	
$\cos\omega t$	$\frac{s}{(s^2+\omega^2)(s^2+4)} = \left(\frac{s}{s^2+\omega^2} - \frac{s}{s^2+4}\right)\frac{1}{4-\omega^2}$	$\frac{\cos\omega t - \cos 2t}{4-\omega^2}$; $(\omega \neq$	
	$\frac{s}{(s^2+4)^2}$ $(\omega = 2)$	$-t\sin 2t$ $(\omega = 2)$	
$1-u_a(t)$ 	$\frac{1-e^{-as}}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\left(1-e^{-as}\right)$	$\frac{1-\cos 2t}{4} - \frac{1-\cos 2(t-a)}{4}u_a$	 

Same problem for damped oscillator, with several sources f(t)

$$\begin{cases} y'' + 2y' + 5y = f(t) \\ y(0) = A \\ y'(0) = B \end{cases} \longrightarrow Y = \frac{F(s)}{s^2+2s+5} + \frac{As + (B+2A)}{s^2+2s+5} = \frac{A(s+1) + (B+A)}{(s+1)^2+4}$$

$$\frac{F(s)}{s^2+2s+5} \to y_p \qquad \frac{A(s+1)+(B+A)}{(s+1)^2+4} \to e^{-t}\left(A\cos 2t + \frac{B+A}{2}\sin 2t\right)$$

f	Y_p	y_p	
1	$\frac{1}{s(s^2+2s+5)} = \frac{1}{5}\left(\frac{1}{s} - \frac{s+2}{(s+1)^2+4}\right)$	1/2*sin(2t) $\frac{1 - e^{-t}(\cos 2t + \cancel{\sin 2t})}{5} = z(t)$	
e^{-3t}	$\frac{1}{(s+3)(s^2+\dots)} = \frac{1}{8}\left(\frac{1}{s+3} - \frac{s-1}{(s+1)^2+4}\right)$	$\frac{e^{-3t} - e^{-t}(\cos 2t - \frac{1}{2}\sin 2t)}{8}$	
u_a	$\frac{e^{-as}}{s(s^2+\dots)} = \frac{e^{-as}}{5}\left(\frac{1}{s} - \frac{s+2}{(s+1)^2+4}\right)$	$z(t-a)u_a$ - shifted z	
$\cos\omega t$	$\frac{1}{(s^2+\omega^2)(s^2+2s+5)} = \frac{1}{(\omega^4-6\omega^2+25)}\left[\frac{2s+(\omega^2-5)}{s^2+\omega^2} + \frac{2s+(\omega^2-1)}{(s+1)^2+4}\right]$	$\frac{\left(2\cos\omega t + \frac{\omega^2-5}{\omega}\sin\omega t\right)}{\omega^4-6\omega^2+25} + \frac{e^{-t}\left(2\cos 2t + \frac{\omega^2-1}{2}\sin 2t\right)}{\omega^4-6\omega^2+25}$	

Convolution & fundam. solution of IVP

Convolution: $(f*g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$

Basic property: $\boxed{f*g \;\underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}}\; F(s)\, G(s)}$

Application to IVP:

$$\begin{cases} L[y] = f(t) \\ y(0) = 0 \\ \dots \quad \dots \end{cases} \longrightarrow Y(s) = \frac{1}{p(s)} F(s) \xrightarrow{\mathcal{L}^{-1}} \boxed{y(t) = (K*f)(t)}$$

Fundamental solution of IVP:

$$K(t) = \mathcal{L}^{-1}\left(\frac{1}{p(s)}\right) = \dots$$

L	$p(\lambda)$	K
$D+a$	$\lambda+a$	e^{-at}
D^2+a^2	λ^2+a^2	$\frac{\sin at}{a}$
D^2+aD+b	$\lambda^2+a\lambda+b$, $\lambda_{1,2}$ real	$\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}$
	$\lambda = -\alpha \pm i\beta$	$e^{-\alpha t}\frac{\sin\beta t}{\beta}$