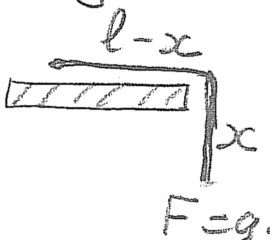
Applications of LDS

Examples:

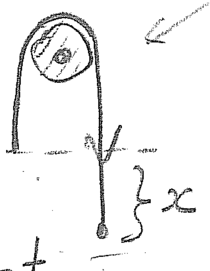
1. Sliding chain.



$$\boxed{\ell \ddot{x} = g x}$$

Cable over frictionless peg

Chain on cogwheel



$$\boxed{\left(\ell + \frac{m}{2}\right)\ddot{x} = g\ell}$$

m = mass (wheel

DS

Direct solution $x(t) = \dots$

$$\boxed{\begin{cases} \dot{x} = v \\ \dot{v} = \frac{g}{\ell} v \end{cases}} \quad A = \begin{bmatrix} 0 & 1 \\ g/\ell & 0 \end{bmatrix} \Rightarrow \begin{array}{c|c} \alpha = \sqrt{g/\ell} & -\alpha \\ \hline \begin{pmatrix} 1 \\ \alpha \end{pmatrix} & \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} \end{array} \Rightarrow \boxed{\underline{Y}(t) = c_1 e^{\alpha t}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} + c_2 e^{-\alpha t}\begin{pmatrix} 1 \\ -\alpha \end{pmatrix}}$$

For IVP: $\begin{cases} x(0) = b > 0 \\ v(0) = 0 \end{cases} \Longrightarrow \underline{Y}(t) = \frac{b}{2}\left[e^{\alpha t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-\alpha t}\begin{pmatrix} 1 \\ -\alpha \end{pmatrix}\right]$

$$x(t) = \frac{b}{2}\left(e^{\alpha t} + e^{-\alpha t}\right) = b \cosh \alpha t$$

Sliding time: $\boxed{x(\overset{?}{t_s}) = \ell} \Rightarrow \boxed{t_s = \cosh^{-1}\left(\frac{\ell}{b}\right) = \frac{1}{\alpha}\ln\left(\frac{\ell + \sqrt{\ell^2 - b^2}}{b}\right.}$

Problem 1: i) Solve sliding chain problem for IV $x(0) = 0$; $v(0) = v_0 > 0$. Find sliding time t_s.

ii) Compute solution $\geq t_s$ for $\ell = 5m$; $v_0 = .5\ m/s$ ($g = 10\ m/s^2$)

Transport (drug delivery)

$$①\ \to \boxed{GI} \xrightarrow[a]{\text{Absorp.}} \boxed{BC} \xrightarrow[k]{\text{Clear}}$$

$g(t)$ $b(t)$ – levels in GI & BC

$$\text{LDS}\begin{cases}\dot g = -ag \\ \dot b = ag - kb\end{cases} \Bigg|\ g(0) = g_0 - \underline{\text{Dose}}$$

$$A = \begin{bmatrix} -a & 0 \\ a & -k \end{bmatrix}$$

$$\begin{array}{c|c} -a & -k \\ \hline \begin{pmatrix} k-a \\ a \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array}$$

Direct: $g(t) = g_0 e^{-at}$

$$\boxed{b(t) = \frac{a}{k-a}\left(e^{-at} - e^{-kt}\right) g_0} \qquad (1)$$

Problem 2: i) Show that maximal b-level

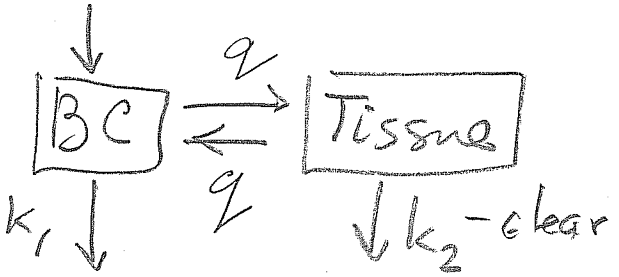
$$b_{max} = (\alpha - 1)\,\alpha^{\alpha/1-\alpha}\, g_0 \quad \left(\alpha = \frac{k}{a}\right) \text{ at } t_{max} = \frac{1}{k-a} \ln \frac{k}{a}.$$

ii) Compute & plot solution $b(t)$ for $a = .02/\text{min}$, $k = .05/\text{min}$.

iii) Find dose d_0 required to get $b_{max} > 5$ mg

Hint: change variable $t \to u = e^{-at}$ in (i

② IV drug;



Volumes: V_1 (BC)

V_2 (Tis)

Concentrations: $x(t)$ - BC

$y(t)$ - Tis

$$2)\begin{cases} V_1 \dot{x} = q(y-x) - k_1 x \\ V_2 \dot{y} = q(x-y) - k_2 y \end{cases} \quad \begin{array}{l} x(0) = D_0 \text{ - dose} \\ y(0) = 0 \end{array}$$

Problem 3: Solve (2) for $V_1 = 3L$; $V_2 = 2L$

$q = .5\ L/min$; $k_1 = .05/min$; $k_2 = .03/min$

Prot $\{x(t), y(t)\}$ Find y_{max} .

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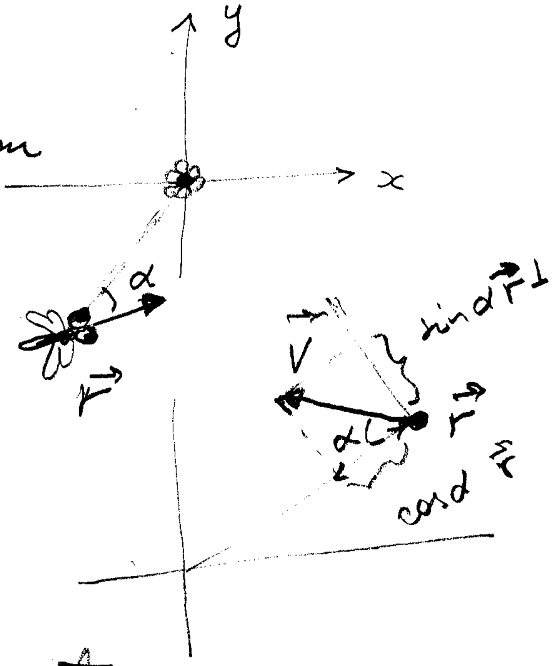
Bug flight

Bug's eye is fixed on object at a fixed angle $0<\alpha<\pi/2$ so velocity at any position $\vec r=(x,y)$ is given by



$$\boxed{\vec V(\vec r)=\frac{V_0}{r}\left(-\cos\alpha\,\vec r+\sin\alpha\,\vec r^\perp\right)}$$

Here V_0 – const speed; $r=|\vec r|$

$\vec r=(-y,x)$ – perpendicular vector.

The eq-ns of motion: $\dot{\vec r}=\vec V(\vec r)$

turns into ODS

$$\boxed{\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}=\frac{V_0}{\sqrt{x^2+y^2}}\begin{bmatrix}-\cos\alpha & -\sin\alpha\\ \sin\alpha & -\cos\alpha\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}}$$

Scalar factor $\frac{V_0}{r}$ does not affect trajectories, given by the linear ODS with matrix

$A=\begin{bmatrix}-c_\alpha & -s_\alpha\\ s_\alpha & -c_\alpha\end{bmatrix}$ ⇒ complex eigenvalues

$\boxed{\lambda_{1,2}=-c_\alpha\pm is_\alpha}$ ⇒ spiral sink