# Summary: Models

1st order DE

## I. Population growth:

$P(t)$ - popul. (state)

$$\frac{dP}{dt} = B - D \quad \text{(birth rate - death rate)}$$

(a) simple (linear): $B = bP$, $D = dP$

$$\boxed{\frac{dP}{dt} = \underbrace{(b-d)}_{\alpha} P}$$

$\alpha = [1/\text{year}]$ - growth rate/capita

(b) logistic:

$$\boxed{\frac{dP}{dt} = \alpha P(1 - P/N)} = f(P)$$

$\alpha$: growth rate; $N$: carrying capacity

(c) growth vs. harvest:

$$\boxed{P' = f(P) - h}$$

$h$: harvest rate

## II. Mech. / motion:

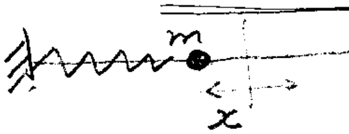
$m \to F$, at $x$

$\left\{\begin{array}{l} x \text{ - position} \\ \dot{x} = v \text{ - velocity} \\ \ddot{x} = \dot{v} = a \text{ - accel.} \end{array}\right\}$ state

Newton:

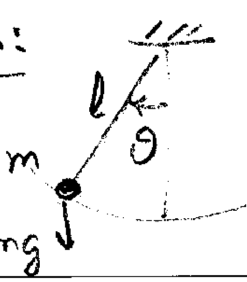
$$\underbrace{\boxed{m\ddot{x} = F(x, \dot{x}, t)}}_{2^{\text{nd}}\ \text{DE}} \Leftrightarrow \underbrace{\boxed{\begin{cases} \dot{x} = v \\ \dot{v} = F(\ldots \end{cases}}}_{DS}$$

• oscillator:



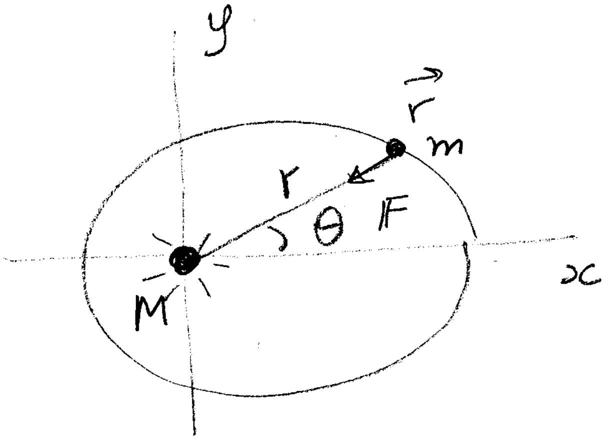

$$\boxed{m\ddot{x} = -kx} \Leftrightarrow \boxed{\begin{array}{l} \dot{x} = v \\ \dot{v} = -\frac{k}{m}x \end{array}}$$

pendulum:




$$\boxed{l\ddot{\theta} = -g\sin\theta} \Leftrightarrow \boxed{\begin{array}{l} \dot{\theta} = \omega \\ \dot{\omega} = -\frac{g}{l}\sin\theta \end{array}}$$

# Kepler (planetary) 2-body system:




## Full system:

position: $\vec{r} = (x, y)$

velocity: $\vec{V} = (\dot{x}, \dot{y})$

accel: $\vec{a} = (\ddot{x}, \ddot{y})$

force of gravity: $\boxed{F(\vec{r}) = -\dfrac{GMm}{r^2}\dfrac{\vec{r}}{r}}$

inverse squar-

(1) $\boxed{m\ddot{\vec{r}} = -F(\vec{r})}$

Coupled 2nd order DS: (1') $\boxed{\begin{aligned} \ddot{x} &= -\frac{GMx}{(x^2+y^2)^{3/2}} \\ \ddot{y} &= -\frac{GMy}{(x^2+y)^{3/2}} \end{aligned}}$

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## Reduced system for $r = r(\theta)$

(radius = function of θ!)

(2) $\boxed{\dfrac{dr}{d\theta} = \dfrac{r}{J}\sqrt{2(Er^2 + GMr) - J^2} = f(r, E, J)}$

1st order ODE with 2 constants (conserved integrals):

$E = \dfrac{mV^2}{2} - \dfrac{GMm}{r}$ – total energy

$J = r^2\dot{\theta}$ – angular momentum

## III. Population dynamics (competing species): R - rabbits, F - foxes

growth rate

$$\begin{cases} R' = a R - c R F \\ F' = d R F - e F \end{cases}$$

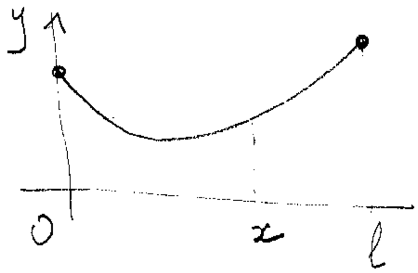
killing rate

attrition

profession due to killing

## IV. Geometry

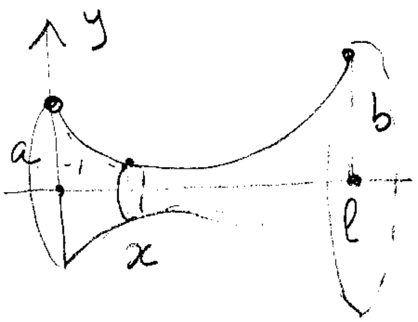
1) Heavy chain: $y(x)$ - ?




$$\boxed{y' = \sqrt{\left(\frac{y-a}{b}\right)^2 - 1}}$$

a, b - param. (to be determ)

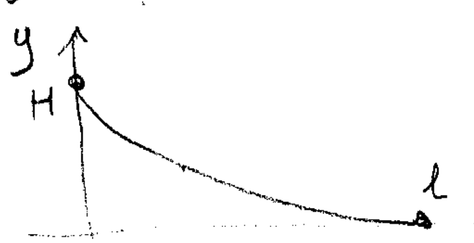
2) Minimal surface of revolution: $y(x)$ - ?




$$\boxed{y' = \sqrt{\left(\frac{y}{c}\right)^2 - 1}}$$

c - param.

3) Fastest slide: $y(x)$ - ?




$$\boxed{y' = \sqrt{\frac{c+y}{H-y}}}$$

c - param.

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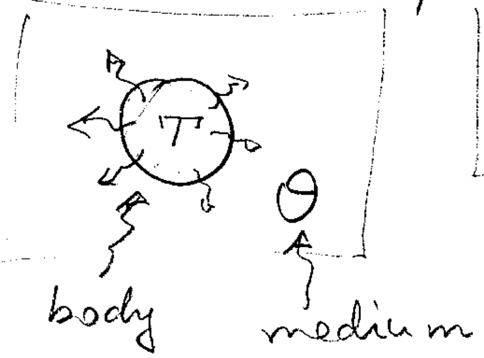
# More models

## ① Heating/cooling (Newton, Fourier)

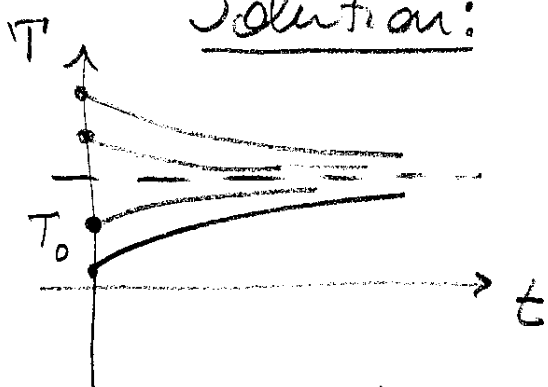
Two bodies or body & medium exchange heat at a rate proportional to the temperature difference:

(i)




$$\boxed{\frac{dT}{dt} = -k(T-\theta)}$$

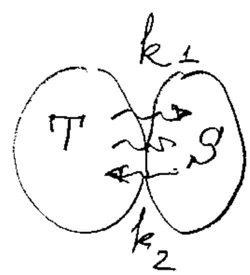
$k = \dfrac{\text{Heat conductivity}}{\text{Heat capacity}}$ - cooling rate




Solution: $\boxed{T(t) = \underset{\text{equilibrium}}{\theta} + c\,e^{-kt}}$ ; $T(0)=T_0$

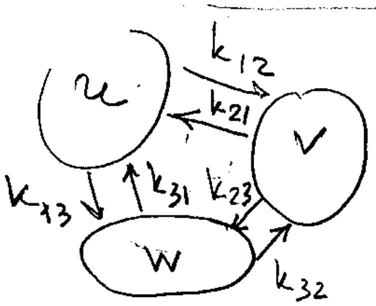
IVP: $\boxed{T(t) = \theta + (T_0-\theta)e^{-kt}}$

(ii) Two bodies:




$$\Rightarrow \begin{cases} T' = -k_1(T-S) \\ S' = -k_2(S-T) \end{cases}$$

**Problem:** Write DE system for 3 bodies:




with temperatures {u(t),v(t),w(t)} and heat exchange coefficients: $\{k_{ij}\}$