DS models

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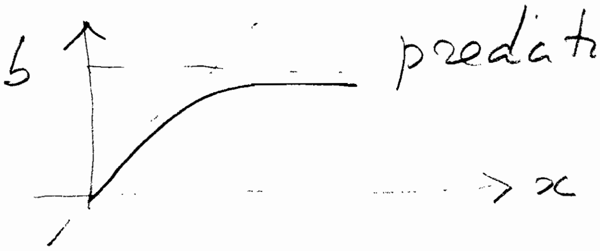
I. Population Biology

Pred-prey

1) $\begin{cases} \dot{x} = ax - bxy - h \\ \dot{y} = cxy - dy \end{cases}$ Volterra-Lotka w. source/harvest

2) $\begin{cases} \dot{x} = a(1 - x/N)x - bxy \\ \dot{y} = cxy - dy \end{cases}$ Logistic prey

3) $\begin{cases} \dot{x} = a(1 - x/N)x - b\dfrac{x}{x+x_0}y \\ \dot{y} = c\dfrac{x}{x+x_0}y - dy \end{cases}$ Predation w. satiation



predation f-n

Competion/cooperation

$$\begin{cases} \dot{x} = a_1(1 - x/N_1)x \mp b_1 xy \\ \dot{y} = a_2(1 - x/N_2)y \mp b_2 xy \end{cases}$$

Food chain:

$$\begin{cases} \dot{x} = ax - bxy \\ \dot{y} = cxy - dyz \\ \dot{z} = eyz - fz \end{cases}$$

"y" predates on "x", "z" predates on "y"

Mechanical systems (oscillators)

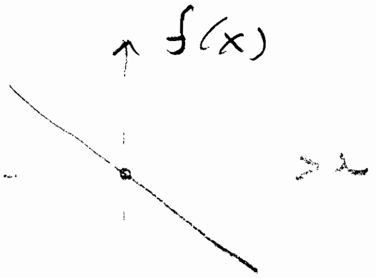
DE $\boxed{m\ddot{x} = f(x) - d(\dot{x}) + b(t)}$ $\Longleftrightarrow$ DS $\begin{cases} \dot{x} = v \\ \dot{v} = \frac{1}{m} \underbrace{F(x, v, t)}_{\text{combined force}} \end{cases}$

potential force, friction, External

Examples:

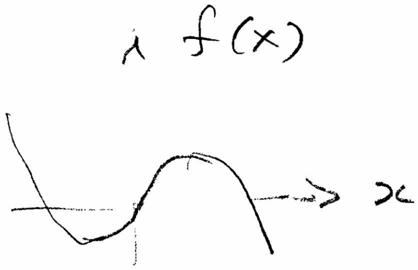
1) Linear oscill: $\boxed{f = -kx}$

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{k}{m} x - \frac{d}{m} v + b(t) \end{cases}$$

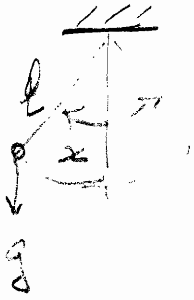


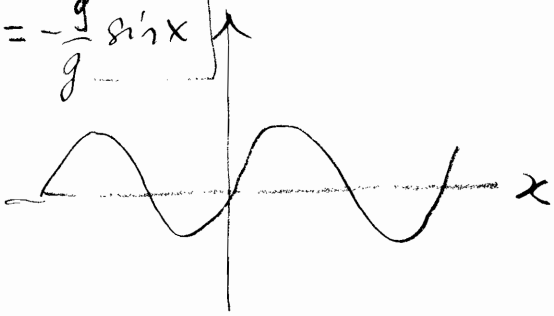
2) Duffing oscill: $\boxed{f = ax - bx^3}$

$$\begin{cases} \dot{x} = v \\ \dot{v} = ax - bx^3 - dv + \dots \end{cases}$$



3) Pendulum: $\boxed{f = -\frac{g}{g} \sin x}$





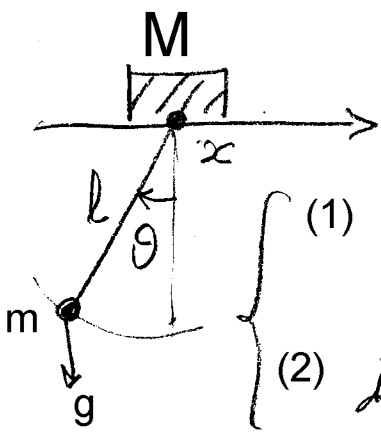
$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{g}{\ell} \sin x - dv + \end{cases}$$

4) Non-lin. x-dependent friction (non-mech.)

$$\begin{cases} \dot{x} = v \\ \dot{v} = -kx + (1 - x^2) v \end{cases}$$

van der Pole

Sliding pendulum.



Equations of motion are based on two conservation laws

$$\begin{cases} (1) \quad \dot{x} - \frac{m}{m+M} l \sin\theta \, \dot{\theta} = p\text{-const} & \text{- horizontal momentum} \\ (2) \quad \frac{d}{dt}\left[\dot{\theta} - \frac{\sin\theta}{l}\dot{x}\right] = -\frac{g}{l}\sin\theta & \text{- angular momentum} \end{cases}$$

For p=0 solve (1) for $\dot{x} = \frac{m}{M+m}\sin\theta\,\dot{\theta}$ to get 2-nd order DE for θ

(Modified pendulum) DE

$$\boxed{\frac{d}{dt}\Big[\underbrace{(1-\gamma\sin^2\theta)\dot{\theta}}_{v}\Big] = -\frac{g}{l}\sin\theta} \qquad \gamma = \frac{m}{M+m}$$

DS:

$$\boxed{\begin{aligned} \dot{\theta} &= \frac{v}{1-\gamma\sin^2\theta} \\ \dot{v} &= -\frac{g}{l}\sin\theta \end{aligned}}$$

$\gamma = 0$ $(M=\infty)$ gives standard pendul.

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

Conserved integral (Hamiltonian).

$$\boxed{h(\theta, v) = \frac{v^2}{2(1-\gamma\sin^2\theta)} - \frac{g}{l}\cos\theta} \quad \text{— "energy"}$$

$$\begin{cases} \dot{\theta} = \frac{\partial h}{\partial v} \\ \dot{v} = -\frac{\partial h}{\partial \theta} \end{cases}$$

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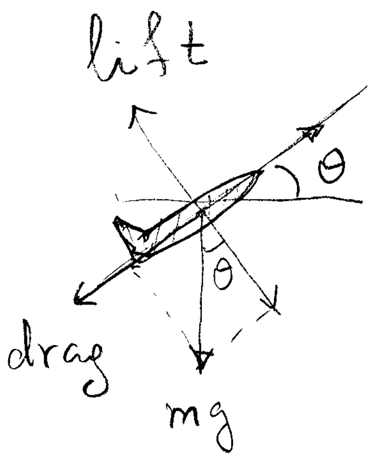
Glider



θ – angle of attack

3 forces: gravity, lift, drag

Assume: drag $= Dv^2$, lift $= Bv^2$
proportional to v^2

Get NL ODS for (v, θ)

$$(1)\quad \begin{cases} \dot{v} = -\frac{D}{m} v^2 - g \sin\theta & \text{– tangent accel.} \\ v\dot{\theta} = \frac{B}{m} v^2 - g\cos\theta & \text{– normal accel.} \end{cases}$$

For $D = 0$ (no drag) get equilibrium

$\theta_c = 0$; $\quad v_c = \sqrt{g\frac{m}{B}}$ – frictionless horiz. flight $\longrightarrow v_c$

Rescale (1) for dimensionless velocity v/v_c

Rescaled eq-ns:

$$(2)\quad \begin{cases} \dot{v} = -A\, v^2 - \sin\theta \\ \dot{\theta} = v - \frac{\cos\theta}{v} \end{cases} \quad \text{where } A = \frac{D}{B} = \frac{\text{drag}}{\text{lift}}$$

Chemical reactions

1. $A + B \underset{k_2}{\overset{k_1}{\rightleftarrows}} C$ - reactants

$x \quad y \quad z$ - concentrations

k_1, k_2 - reaction rates (binding, dissos.)

DS $\begin{cases} \dot{x} = -k_1 xy + k_2 z \\ \dot{y} = -k_1 xy + k_2 z \\ \dot{z} = k_1 xy - k_2 z \end{cases}$

Mass conserv.

$x + z = N$ - const

$y + z = M$ - const

$$\Longrightarrow \boxed{\dot{z} = k_1 (N - z)(M - z) - k_2 z} \quad DE$$

Problem: Show DE (& DS) has stable equilibrium z^*. Find it

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2. Ozone + NO_x photochemistry

$O + O_2 \to O_3$

$NO + O_3 \to NO_2 + O_2$

$NO_2 \xrightarrow{h\nu} NO + O$

Reduced scheme:

$O \xrightarrow{k_0} O_3$

$NO + O_3 \xrightarrow{k_1} NO_2$

$NO_2 \xrightarrow{k_2} NO + O$

O	u	spp/concent
O_3	x	
NO	y	
NO_2	z	

$\begin{cases} \dot{u} = -k_0 u + k_2 z \\ \dot{x} = k_0 u - k_1 y x \\ \dot{y} = -k_1 y x + k_2 z \\ \dot{z} = k_1 xy - k_2 z \end{cases}$

$\boxed{k_0 u \approx k_2 z}$ as $k_0 \gg k_{1,2}$

$\Rightarrow \begin{cases} \dot{x} = k_2 z - k_1 y x \\ \dot{y} = -k_1 xy + k_2 z \\ \dot{z} = k_1 xy - k_2 z \end{cases} \Leftrightarrow x + y \underset{k_2}{\overset{k_1}{\rightleftarrows}} z$

Like system 1

SIR model of epidemics

S - susceptible, I - infected, R - recovered } fractions of population $S+I+R$ assumed const.

Transmission & recovery process

$$(3D)\begin{cases} \dfrac{dS}{dt} = -\alpha SI + R/b; \\ \dfrac{dI}{dt} = \alpha SI - I/c; \\ \dfrac{dR}{dt} = I/c - R/b \end{cases}$$

α - transmission rate

b - mean loss time of immunity

c - mean duration of illness

Total pop. $\frac{d}{dt}(S+I+R)=0; \iff S+I+R=P_0$ - const

$\Rightarrow R = P_0 - I - S$ reduce to 2D system

$$(2D)\begin{cases} \dot{S} = -\alpha SI + (P_0 - I - S)/b \\ \dot{I} = (\alpha S - 1/c) I \end{cases}$$

SI - model

$$(2D)\begin{cases} \dot{S} = -\alpha SI + I/c \\ \dot{I} = \alpha SI - I/c \end{cases} \Rightarrow S + I = P_0 \text{ - const} \Rightarrow$$

Reduced eq-n

$$(1D) \quad \boxed{\dot{S} = \frac{1}{c}(P_0 - S) - \alpha S(P_0 - S)}$$