D. GURARIE

Linear DS

Coord. form $\qquad\qquad$ **Vector form**

$$\begin{cases} \dot{x} = \boxed{ax+by} + \boxed{p} \\ \dot{y} = \boxed{cx+dy} + \boxed{q} \end{cases} \Rightarrow \boxed{\dot{Y} = AY + B};$$

maxtrix of coeff: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Source: $B = \begin{pmatrix} p \\ q \end{pmatrix}$; state: $Y = \begin{pmatrix} x \\ y \end{pmatrix}$

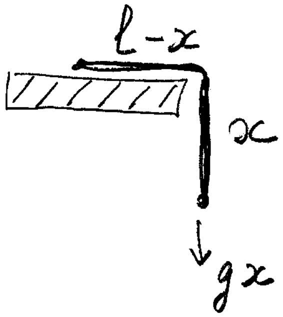
Models, examples

1. **Linear oscill:**

2nd DE: $\boxed{m\ddot{x} = -kx - \alpha\dot{x} + f} \Rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = -\frac{k}{m}x - \frac{\alpha}{m}v + \frac{f}{m} \end{cases}$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\alpha}{m} \end{bmatrix}; \quad B = \begin{pmatrix} 0 \\ f/m \end{pmatrix}$$

2. **Sliding chain w/o friction:**



$$\underbrace{l}_{\text{mass}}\ddot{x} = gx \Rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = \frac{g}{l}x \end{cases} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ g/l & 1 \end{bmatrix}.$$

3. **Linear compo./cooperation:**

$$\begin{cases} \dot{x} = ax \pm by + p \\ \dot{y} = cx \pm dy + q \end{cases} \Rightarrow A = \begin{bmatrix} a & \pm b \\ \pm c & d \end{bmatrix}; \quad B = \begin{pmatrix} p \\ q \end{pmatrix} \text{ source / harvest}$$

5. Economics (buyer/seller market of prob. 5.1:20-23)

B_0 S_0 - equilibrium values (buyers/sellers)

$(B_0 + b(t),\ S_0 + s(t))$ - market dependent fluctuations

Linearized system: $\begin{cases} \dot{b} = -\alpha b + \beta s \\ \dot{s} = \gamma b - \delta s \end{cases}$

Matrix $A = \begin{bmatrix} -\alpha & \beta \\ \gamma & -\delta \end{bmatrix}$ has coeff. $\pm$ determined by contribution of (b, s) to increase/depression of prices.

---

6. Migration: $\underset{A}{(x)} \underset{qy}{\overset{px}{\rightleftarrows}} \underset{B}{(y)}$ migration pattern

states: A, B

p, q - fractions

Discrete time step model: $t_n = n\,\delta t$

$$\begin{cases} x_{n+1} = (1 - p\,\delta t)x_n + \delta t\, q\, y_n \\ y_{n+1} = p\,\delta t\, x_n + (1 - q\,\delta t) y_n \end{cases} \Rightarrow \boxed{Y_{n+1} = \begin{bmatrix} 1-p\delta t & q\delta t \\ p\,\delta t & 1-q\delta t \end{bmatrix} Y_n}$$

finite-difference

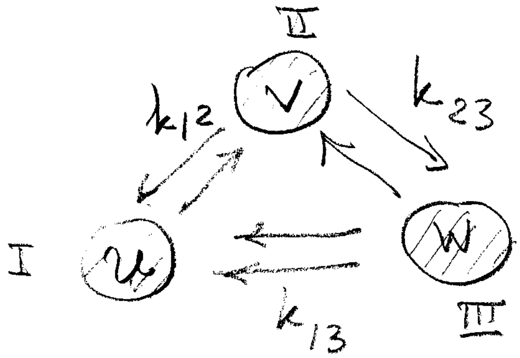
Continuous model:

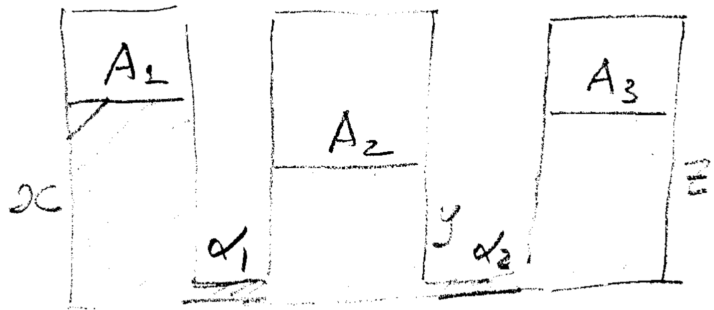
$\delta t \to 0$

$$\begin{cases} \dot{x} = -px + qy \\ \dot{y} = px - qy \end{cases} \Rightarrow \text{matrix } A = \begin{bmatrix} -p & q \\ p & -q \end{bmatrix}$$

7. Heat exchange. Fluid exchange

A_i – cross-sect. areas





3 bodies with temperatures: u, v, w

heat-exchang coeff. (k_{ij})

3 reservoirs with heights x, y, z and pipes exchange rates: α_1 α_2

$$\begin{cases} \dot{u} = k_{12}(v-u) + k_{13}(w-u) \\ \dot{v} = k_{12}(u-v) + k_{23}(w-v) \\ \dot{w} = k_{13}(u-w) + k_{23}(v-w) \end{cases}$$

$$\begin{cases} A_1\dot{x} = \alpha_1(y-x) \\ A_2\dot{y} = \alpha_1(x-y) + \alpha_2(z-y) \\ A_3\dot{z} = \alpha_2(y-z) \end{cases}$$

$$A = \begin{bmatrix} -(k_{12}+k_{13}) & k_{12} & k_{13} \\ k_{12} & -(k_{12}+k_{23}) & k_{23} \\ k_{13} & k_{23} & -(k_{13}+k_{23}) \end{bmatrix}$$

Problem: Write matrix for fluid system.

M224 D. GURARIE

Activator - inhibitor systems

In many biological systems different components affect each other growth/decay

Example: Glucose in blood stimulates production of Insulin; Insulin removes Glucose

Marking:
- —■ - activation/stimulation
- —□ - inhibition/removal
- ↻ - growth
- - - → - decay

		Glucose - Insulin
$a\circlearrowright x \xrightarrow{c}\blacksquare\, y \overset{d}{\dashrightarrow}$, $\;x\ \blacksquare\!\xleftarrow{b} y$	$\circlearrowright x \longrightarrow\square\, y \circlearrowleft$, $\;x\ \blacksquare\!\longleftarrow y$	$S \to x \longrightarrow\blacksquare\, y \dashrightarrow$, $\;x\ \square\!\longleftarrow y$
$\begin{cases}\dot x = ax+by\\ \dot y = cx-dy\end{cases}$ $\begin{bmatrix}+ & +\\ + & -\end{bmatrix}$	$\begin{cases} ax+by\\ -cx+dy\end{cases}$	$\begin{cases}\dot x = S-by\\ \dot y = cx-dy\end{cases}$
$A=\begin{bmatrix}a & b\\ c & -d\end{bmatrix}$	$\begin{bmatrix}a & b\\ -c & d\end{bmatrix}$	$\begin{bmatrix}0 & -b\\ c & -d\end{bmatrix}\cdot Y+\begin{pmatrix}S\\0\end{pmatrix}$
$\begin{cases} tr = a-d \gtrless 0\\ det = -(ad+bc)<0\end{cases}$ saddle	$tr = a+d>0$, $det = ad+bc>0$ unst.	$tr=-d<0$, $det = bc>0$ stable

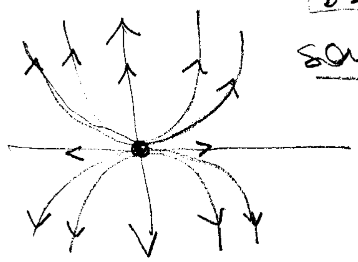
Solutions:

1) Diagonal matrix

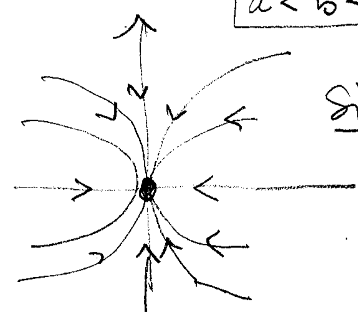
$$\left\{\begin{array}{l}\dot{x} = ax\\ \dot{y} = by\end{array}\right. \Rightarrow A=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix} \Rightarrow \underline{Y}(t) = \begin{pmatrix}c_1e^{at}\\ c_2e^{bt}\end{pmatrix} = \underbrace{c_1\underbrace{\boxed{e^{at}\begin{pmatrix}1\\0\end{pmatrix}}}_{\underline{Y}_1(t)} + c_2\underbrace{\boxed{e^{bt}\begin{pmatrix}0\\1\end{pmatrix}}}_{\underline{Y}_2(t)}}_{\text{Fund. pair}}$$

Cases:

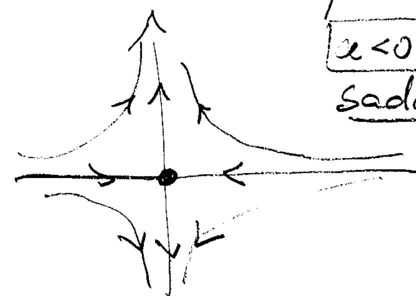
$\boxed{b>a>0}$ source



$\boxed{a<b<0}$ sink

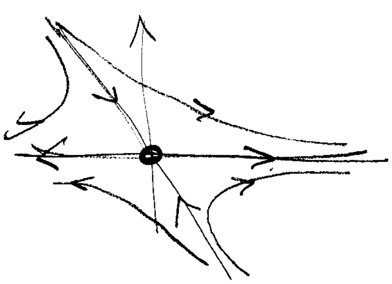


$\boxed{a<0<b}$ saddle



2) $$\left\{\begin{array}{l}\dot{x} = 2x+y\\ \dot{y} = -3y\end{array}\right. \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}c_1e^{2t} - \frac{c_2}{5}e^{-3t}\\ c_2e^{-3t}\end{pmatrix}$$

$$\underline{Y} = c_1\underbrace{\boxed{\begin{pmatrix}1\\0\end{pmatrix}e^{2t}}}_{\underline{Y}_1} + c_2\underbrace{\boxed{\begin{pmatrix}-1/5\\1\end{pmatrix}e^{-3t}}}_{\underline{Y}_2} \text{ – fund. pair}$$



IVP solution for $\underline{Y}(0) = \begin{pmatrix}1\\4\end{pmatrix}$: ...

Linear superposition:

any solution of $\underline{Y}' = A\underline{Y}$ is a linear combination of fund. pair (set) $\underline{Y}_1(t)$, $\underline{Y}_2(t)$

3) Oscill: $\ddot{x} = -4x \rightarrow \underline{Y}_1 = \begin{pmatrix}\cos 2t\\ -2\sin 2t\end{pmatrix}$; $\underline{Y}_2 = \begin{pmatrix}\sin 2t\\ 2\cos 2t\end{pmatrix}$