Multiplier method

Multiplier f-n for DE: $\boxed{y' + a(t)y = b}$

$$\mu(t) = e^{\int^t a(s)ds} \text{ solves } \boxed{\mu' = a\mu}$$

and $\boxed{\left(\frac{1}{\mu}\right)' = -a\frac{1}{\mu}}$ homog.

GS: $$\boxed{y(t) = \frac{c}{\mu(t)} + \frac{1}{\mu(t)}\int_{t_0}^{t} \mu(s)\, b(s)\, ds} \quad (1)$$

IVP: $$\boxed{y(t) = \frac{y_0}{\mu(t)} + \frac{1}{\mu(t)}\int_{t_0}^{t} \mu(s)\, b(s)\, ds} \quad (2)$$

with $\mu(t) = e^{\int_{t_0}^{t} a(s)\, ds}$

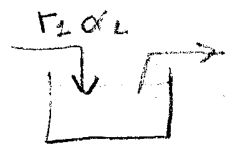
For const $a \Rightarrow \boxed{\mu(t) = e^{at}}$ and

(1)–(2) are convolution integrals:

$$y(t) = \begin{cases} c e^{-at} + \int_0^t e^{-a(t-s)} b(s) ds \\ y_0 e^{-at} + \underbrace{\int_0^t e^{-a(t-s)} b(s) ds}_{\boxed{e^{-at} * b}} \end{cases}$$

Application to mixing problem

2. Mixing:



$$V = V_0 + (r_1 - r_2)t$$

$$\boxed{q' + \frac{r_2}{V} q = \alpha_1 r_1}$$

$$\mu = e^{\int r_2/V \, dt} = V^{r_2/(r_1 - r_2)} = V^{\beta}$$

Particular: $q_p = \frac{1}{\mu} \int \mu \alpha_1 r_1 = \frac{\alpha_1 r_1}{V^\beta} \int V^\beta = \frac{\alpha_1 r_1}{\beta + 1} \frac{V^{\beta+1}}{(r_1 - r_2)} = \alpha_1 V$

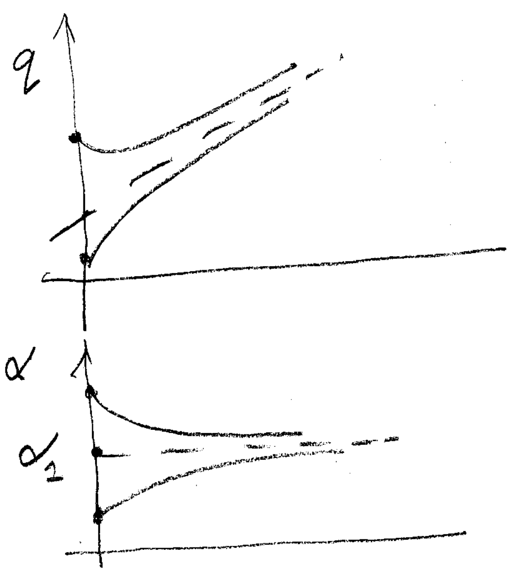
GS: $q = c V^{-\beta} + \alpha_1 V$

IVP: $c = (q_0 - \alpha_1 V_0) V_0^{\beta} \Rightarrow \boxed{q = \alpha_1 V + (q_0 - \alpha_1 V_0)\left(\frac{V_0}{V}\right)^{\beta}}$

$r_1 = 3;\ r_2 = 1;\ \alpha_1 = .05$

$V = V_0 + 2t;\quad \beta = 1/2$

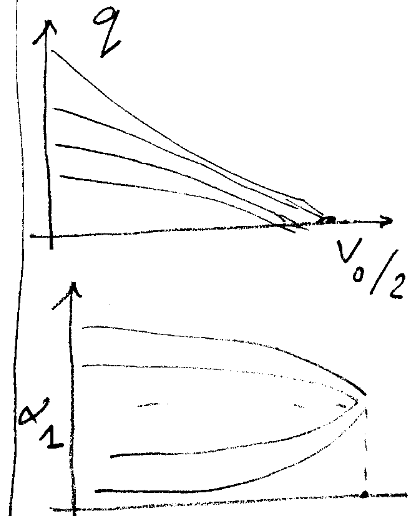
$$q = .05(V_0 + 2t) + (q_0 - .05 V_0)\left(\frac{V_0}{V_0 + 2t}\right)^{1/2}$$



$r_1 = 1;\ r_2 = 3;\ \alpha_1 = .5$

$V = V_0 - 2t;\quad \beta = -3/2$

$$q = .5(V_0 - 2t) + (\ldots)\left(\frac{V_0 - 2t}{V_0}\right)^{3/2}$$



D. GURARIE

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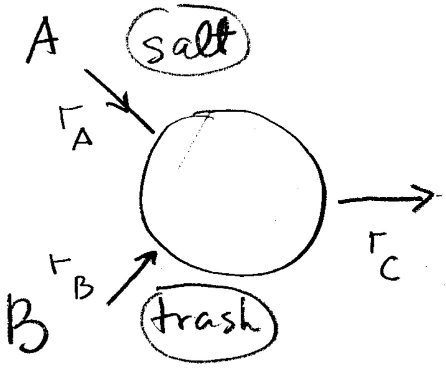
Polluted pond



$r_A = \underbrace{500}\ m^3/\text{day}$; $\alpha_A = .005\ kg/m^3$ - salt

$r_B = \underbrace{750\ m^3/\text{day}}_{\text{water}} + \underbrace{50\ m^3/\text{day}}_{\text{trash}}$

$r_C = 1300\ m^3/\text{day}$ (to keep level)

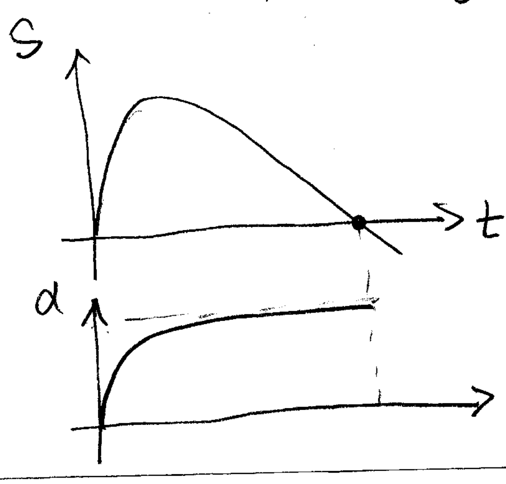
$\underline{V_0 = 10^4\ m^3}$; $S = \#$ salt [kg]

$$\frac{dS}{dt} = \alpha_A r_A + \alpha_B r_B - \alpha_C r_\alpha$$

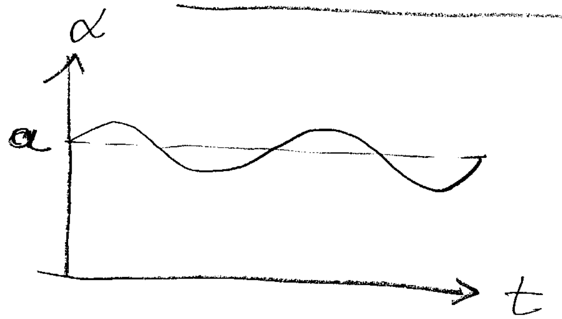
$$\boxed{\frac{dS}{dt} = 5^{-3} \cdot 500 - \frac{S}{V_0 - 50t}\ 1300}$$

$$\begin{cases} S' + \frac{26}{200-t} S = 2.5 \\ S(0) = 0 \end{cases} \Rightarrow \quad \mu = (200-t)^{-26}$$

$$S = \frac{1}{\mu}\ 2.5 \int_0^t \mu(s)\,ds = \frac{200-t}{10} - 20\left(1 - t/200\right)^{26}$$



Loan with fluctuating interest



$$\boxed{y' = \alpha y - p}$$

interest payment

$$\alpha(t) = a + b\cos\omega t$$

multiplier: $\mu(t) = e^{\int_0^t \alpha(\tau)\,d\tau} = e^{\left(at + \frac{b}{\omega}\sin\omega t\right)}$

Solution:

$$y(t) = \mu(t)\left[y_0 - p\int_0^t \frac{d\tau}{\mu(\tau)}\right] - ?$$

Approximations for $\frac{b}{\omega} << 1$

(1) $\mu(t) \approx e^{at}\left(1 + \frac{b}{\omega}\sin\omega t\right)$; $\frac{1}{\mu(t)} \approx e^{-at}\left(1 - \frac{b}{\omega}\sin\omega t\right)$

Problem: For fluctuating interest $\alpha(t)$

(i) Compute approximate solution based on approximate multiplier (1) (You can use mathematica for intergration !)

(ii) Take $a = .05$, $b = .01$, $\omega=1$. plot solutions (mathematica or matlab) for different p and find approximately minimal payment rate.