EXERCISES FOR SECTION 2.3

In Exercises 1–4, we consider the system

$$\frac{dx}{dt} = 2x + 2y$$
$$\frac{dy}{dt} = x + 3y.$$

For the given functions $\mathbf{Y}(t) = (x(t), y(t))$, check to see if $\mathbf{Y}(t)$ is a solution to the system.

1. $(x(t), y(t)) = (2e^t, -e^t)$
2. $(x(t), y(t)) = (3e^{2t} + e^t, -e^t + e^{4t})$
3. $(x(t), y(t)) = (2e^t - e^{4t}, -e^t + e^{4t})$
4. $(x(t), y(t)) = (4e^t + e^{4t}, -2e^t + e^{4t})$

In Exercises 5–12, we consider the partially decoupled system

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = -y.$$

5. Although we can use the method described in this section to derive the general solution to this system, why should we immediately know that $\mathbf{Y}(t) = (x(t), y(t)) = (e^{2t} - e^{-t}, e^{-2t})$ is *not* a solution to the system?

6. Although we can use the method described in this section to derive the general solution to this system, is there an easier way to show that $\mathbf{Y}(t) = (x(t), y(t)) = (4e^{2t} - e^{-t}, 3e^{-t})$ is a solution to the system?

7. Use the method described in this section to derive the general solution to this system.

8. **(a)** Can you choose constants in the general solution obtained in Exercise 7 that yield the function $\mathbf{Y}(t) = (e^{-t}, 3e^{-t})$?
   (b) Suppose that the result of Exercise 7 was not immediately available. How could you tell that $\mathbf{Y}(t) = (e^{-t}, 3e^{-t})$ is not a solution?

9. **(a)** Using the result of Exercise 7, determine the solution that satisfies the initial condition $\mathbf{Y}(0) = (x(0), y(0)) = (1, 0)$.
   (b) In the xy-phase plane, plot the solution curve associated to this solution.
   (c) Plot the corresponding $x(t)$- and $y(t)$-graphs.

10. **(a)** Using the result of Exercise 7, determine the solution that satisfies the initial condition $\mathbf{Y}(0) = (x(0), y(0)) = (-1, 3)$.

(b) In the xy-phase plane, plot the solution curve associated to this solution.

(c) Plot the corresponding $x(t)$- and $y(t)$-graphs.

11. (a) Using the result of Exercise 7, determine the solution that satisfies the initial condition $\mathbf{Y}(0) = (x(0), y(0)) = (0, 1)$.

(b) Using HPGSystemSolver, plot the corresponding solution curve in the xy-phase plane and compare the result with the curve that you would have drawn directly from the direction field for the system.

(c) Using only the solution curve, sketch the $x(t)$- and $y(t)$-graphs.

(d) Compare your sketch with the $x(t)$- and $y(t)$-graphs that HPGSystemSolver plots.

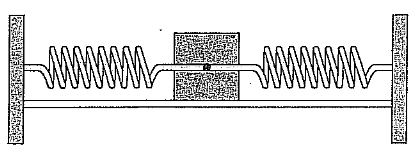
12. (a) Using the result of Exercise 7, determine the solution that satisfies the initial condition $\mathbf{Y}(0) = (x(0), y(0)) = (1, -1)$.

(b) Using HPGSystemSolver, plot the corresponding solution curve in the xy-phase plane and compare the result with the curve that you would have drawn directly from the direction field for the system.

(c) Using only the solution curve, sketch the $x(t)$- and $y(t)$-graphs.

(d) Compare your sketch with the $x(t)$- and $y(t)$-graphs that HPGSystemSolver plots.

In Exercises 13 and 14, we consider a mass sliding on a frictionless table between two walls that are 1 unit apart and connected to both walls with springs, as shown below.



Let k_1 and k_2 be the spring constants of the left and right spring, respectively, let m be the mass, and let b be the damping coefficient of the medium the spring is sliding through. Suppose L_1 and L_2 are the rest lengths of the left and right springs, respectively.

13. Write a second-order differential equation for the position of the mass at time t. [*Hint*: The first step is to pick an origin, that is, a point where the position is 0. The left-hand wall is a natural choice.]

14. (a) Convert the second-order equation of Exercise 13 into a first-order system.

(b) Find the equilibrium point of this system.

(c) Using your result from part (b), pick a new coordinate system and rewrite the system in terms of this new coordinate system.

(d) How does this new system compare to the system for a damped harmonic oscillator?

In Exercises 15–18, a second-order equation for $y(t)$ is given.

(a) Plot its direction field in the yv-plane, where $v = dy/dt$.

(b) Using the guess-and-test method described in this section, find two nonzero solutions that are not multiples of one another.

(c) For each solution, plot both its solution curve in the yv-plane and its $y(t)$- and $v(t)$-graphs.

15. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - 10y = 0$

16. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 0$

17. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + y = 0$

18. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} - 2y = 0$

19. Consider the partially decoupled system

$$\frac{dx}{dt} = 2x - 8y^2$$
$$\frac{dy}{dt} = -3y.$$

(a) Derive the general solution.

(b) Find the equilibrium points of the system.

(c) Find the solution that satisfies the initial condition $(x_0, y_0) = (0, 1)$.

(d) Use HPGSystemSolver to plot the phase portrait for this system. Identify the solution curve that corresponds to the solution with initial condition $(x_0, y_0) = (0, 1)$.

2.4 EULER'S METHOD FOR SYSTEMS

Many of the examples in this chapter include some type of plot of solutions, either as curves in the phase plane or as $x(t)$- or $y(t)$-graphs. In most cases these plots are provided without any indication of how we obtain them. Occasionally the solutions are line segments or circles or ellipses, and we are able to verify this analytically. But more often the solutions do not lie on familiar curves. For example, consider the predator-prey type system

$$\frac{dx}{dt} = 2x - 1.2xy$$
$$\frac{dy}{dt} = -y + 1.2xy$$

unnerving. If the initial condition is sufficiently close to zero, then the solution spirals toward the origin as in the case of our original model. If the initial condition is sufficiently far from the origin, however, the behavior is quite different. The solution in the phase plane moves away from the origin. Solutions with these two types of behavior are separated by solution curves that tend to the equilibrium points as t increases.

The interpretation of the behavior of these solutions in terms of the behavior of the building yields dramatic results. For small oscillations the building sways with decreasing amplitude and eventually returns to its rest position. However, if the initial displacement exceeds a threshold distance, then the amplitude of the solution quickly moves away from zero. The building sways more and more violently, and the result is a disaster.

Reality Check

We must emphasize that this model is only a caricature of the actual dynamics of a swaying building. However, the model does teach an important lesson. Solutions with initial conditions in one region of the phase plane may behave very differently from solutions in another region. The Uniqueness Theorem guarantees that, if an initial condition is in one of these regions, then the corresponding solution stays in the region for all time. The transition between different types of solutions can occur abruptly as the initial conditions are varied. Just because a physical system is "stable" with respect to small initial displacements does not imply that it will be stable with respect to all initial conditions. If this simple model can behave in such a radical way, then we should not be surprised to find such bizarre behavior in an actual building.

EXERCISES FOR SECTION 2.4

1. For the system

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x,$$

the curve $\mathbf{Y}(t) = (\cos t, \sin t)$ is a solution. This solution is periodic. Its initial position is $\mathbf{Y}(0) = (1, 0)$, and it returns to this position when $t = 2\pi$. So $\mathbf{Y}(2\pi) = (1, 0)$ and $\mathbf{Y}(t + 2\pi) = \mathbf{Y}(t)$ for all t.

(a) Check that $\mathbf{Y}(t) = (\cos t, \sin t)$ is a solution.

(b) Use Euler's method with step size 0.5 to approximate this solution, and check how close the approximate solution is to the real solution when $t = 4$, $t = 6$, and $t = 10$.

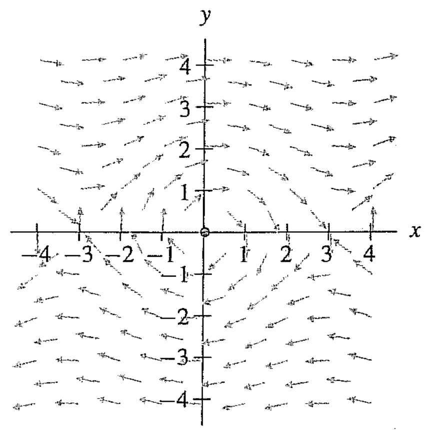
(c) Use Euler's method with step size 0.1 to approximate this solution, and check how close the approximate solution is to the real solution when $t = 4$, $t = 6$, and $t = 10$.

(d) The points on the solution curve $\mathbf{Y}(t)$ are all 1 unit distance from the origin. Is this true of the approximate solutions? Are they too far from the origin or

4. $\dfrac{dx}{dt} = y$

$\dfrac{dy}{dt} = -\sin x$

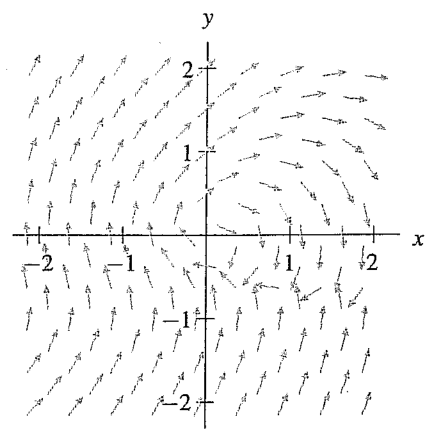
$$\begin{cases} (x_0, y_0) = (0, 2) \\ \Delta t = 0.25 \\ n = 8 \end{cases}$$



5. $\dfrac{dx}{dt} = y + y^2$

$\dfrac{dy}{dt} = -x + \dfrac{y}{5} - xy + \dfrac{6y^2}{5}$

$$\begin{cases} (x_0, y_0) = (1, 1) \\ \Delta t = 0.25 \\ n = 5 \end{cases}$$



6. $\dfrac{dx}{dt} = y + y^2$

$\dfrac{dy}{dt} = -\dfrac{x}{2} + \dfrac{y}{5} - xy + \dfrac{6y^2}{5}$

$$\begin{cases} (x_0, y_0) = (-0.5, 0) \\ \Delta t = 0.25 \\ n = 7 \end{cases}$$