EXERCISES FOR SECTION 1.3

In Exercises 1–6, sketch the slope fields for the given differential equation as follows:

(a) Pick a few points (t, y) with both $-2 \le t \le 2$ and $-2 \le y \le 2$ and plot the associated slope marks without the use of technology.

(b) Use `HPGSolver` to check these individual slope marks.

(c) Make a more detailed drawing of the slope field and then use `HPGSolver` to confirm your answer.

For more details about `HPGSolver` and other programs that are on the CD, see the description of `DETools` inside the front cover of this book.

1. $\dfrac{dy}{dt} = t^2 + t$ **2.** $\dfrac{dy}{dt} = 1 - 2y$ **3.** $\dfrac{dy}{dt} = y + t + 1$

4. $\dfrac{dy}{dt} = t^2 + 1$ **5.** $\dfrac{dy}{dt} = 2y(1 - y)$ **6.** $\dfrac{dy}{dt} = 4y^2$

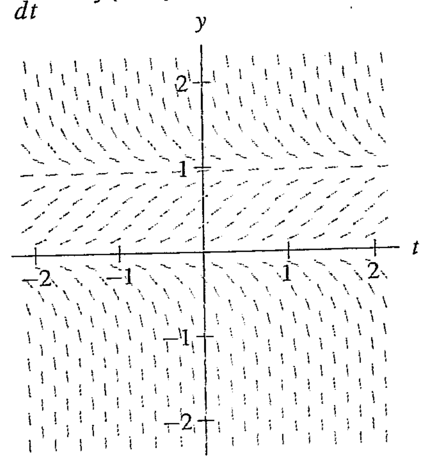
In Exercises 7–10, a differential equation and its associated slope field are given. For each equation,

(a) sketch a number of different solutions on the slope field, and

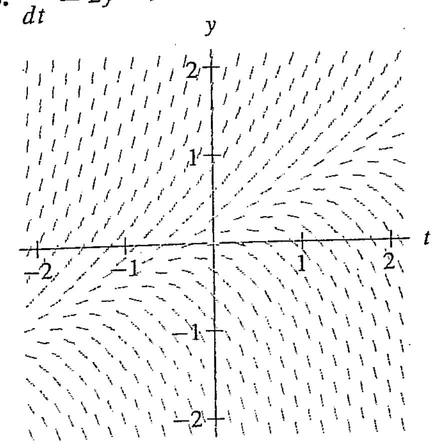
(b) describe briefly the behavior of the solution with $y(0) = 1/2$ as t increases.

You should first answer these exercises without using any technology, and then you should confirm your answer using `HPGSolver`.

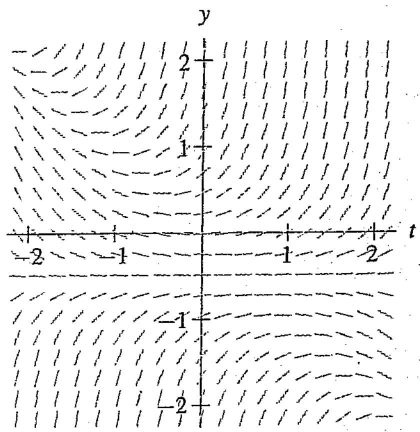
7. $\dfrac{dy}{dt} = 3y(1 - y)$



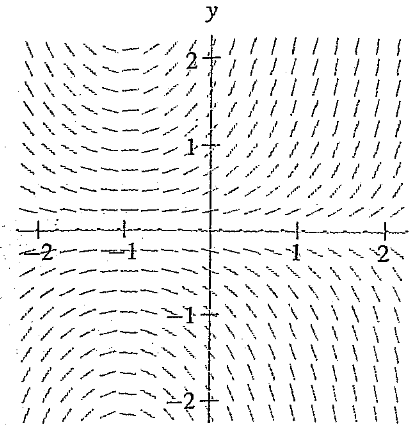
8. $\dfrac{dy}{dt} = 2y - t$



9. $\dfrac{dy}{dt} = \left(y + \frac{1}{2}\right)(y + t)$



10. $\dfrac{dy}{dt} = (t + 1)y$



11. Suppose we know that the function $f(t, y)$ is continuous and that $f(t, 3) = -1$ for all t.

(a) What does this information tell us ~~about~~ the slope field for the differential equation $dy/dt = f(t, y)$?

(b) What can we conclude about solutions $y(t)$ of $dy/dt = f(t, y)$? For example, if $y(0) < 3$, can $y(t) \to \infty$ as t increases?

12. Consider the autonomous differential equation

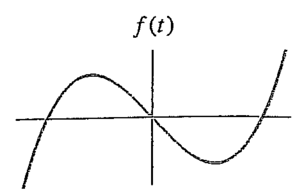
$$\frac{dS}{dt} = S^3 - 2S^2 + S.$$

(a) Make a rough sketch of the slope field without using any technology.

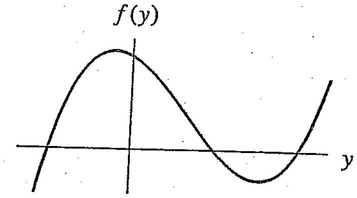
(b) Using this drawing, sketch the graphs of the solutions $S(t)$ with the initial conditions $S(0) = 1/2$, $S(1) = 1/2$, $S(0) = 1$, $S(0) = 3/2$, and $S(0) = -1/2$.

(c) Confirm your answer using `HPGSolver`.

13. Suppose we know that the graph to the right is the graph of the right-hand side $f(t)$ of the differential equation $dy/dt = f(t)$. Give a rough sketch of the slope field that corresponds to this differential equation.



14. Suppose we know that the graph to the right is the graph of the right-hand side $f(y)$ of the differential equation $dy/dt = f(y)$. Give a rough sketch of the slope field that corresponds to this differential equation.

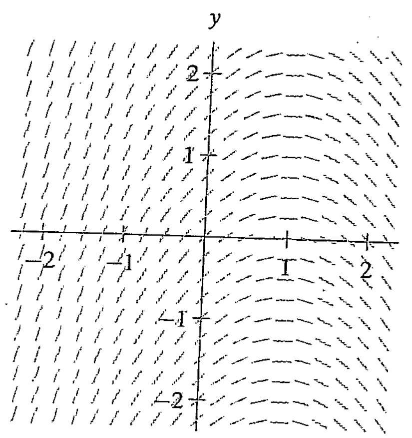


15. Eight differential equations and four slope fields are given below. Determine the equation that corresponds to each slope field and state briefly how you know your choice is correct. You should do this exercise without using technology.

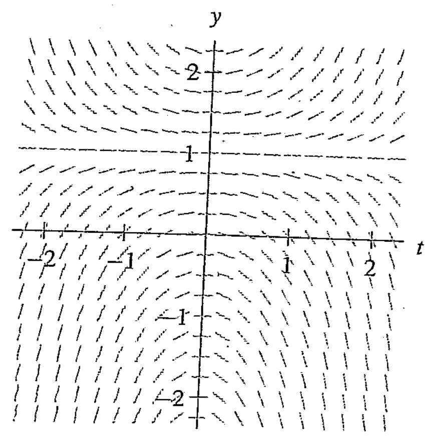
(i) $\dfrac{dy}{dt} = t - 1$ (ii) $\dfrac{dy}{dt} = 1 - y^2$ (iii) $\dfrac{dy}{dt} = y - t^2$ (iv) $\dfrac{dy}{dt} = 1 - t$

(v) $\dfrac{dy}{dt} = 1 - y$ (vi) $\dfrac{dy}{dt} = y + t^2$ (vii) $\dfrac{dy}{dt} = ty - t$ (viii) $\dfrac{dy}{dt} = y^2 - 1$

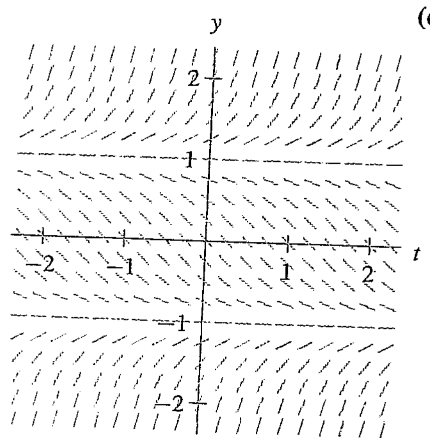
(a)



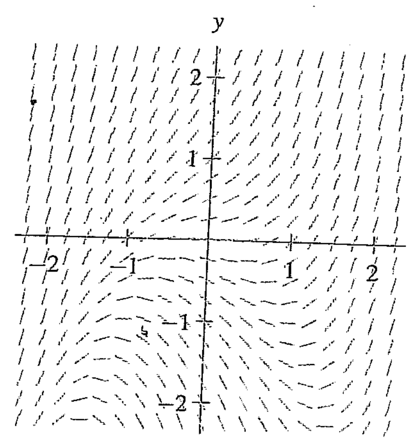
(b)



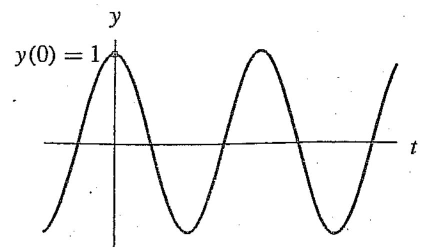
(c)



(d)



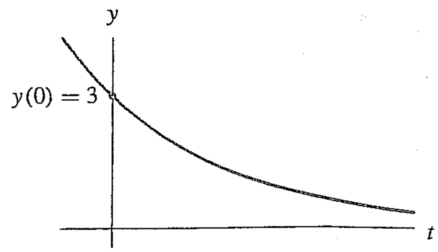
16. Suppose we know that the graph below is the graph of a solution to $dy/dt = f(t)$.



(a) How much of the slope field can you sketch from this information? [*Hint*: Note that the differential equation depends only on t.]

(b) What can you say about the solution with $y(0) = 2$? (For example, can you sketch the graph of this solution?)

17. Suppose we know that the graph below is the graph of a solution to $dy/dt = f(y)$.



(a) How much of the slope field can you sketch from this information? [*Hint*: Note that the equation is autonomous.]

(b) What can you say about the solution with $y(0) = 2$? Sketch this solution.

18. Suppose the constant function $y(t) = 2$ for all t is a solution of the differential equation

$$\frac{dy}{dt} = f(t, y).$$

(a) What does this tell you about the function $f(t, y)$?

(b) What does this tell you about the slope field? In other words, how much of the slope field can you sketch using this information?

(c) What does this tell you about solutions with initial conditions $y(0) \neq 2$?

Exercises 19–23 refer to the RC circuit discussed in this section. The differential equation for the voltage v_c across the capacitor is

$$\frac{dv_c}{dt} = \frac{V(t) - v_c}{RC}.$$

19. Find the formula for the general solution of the RC circuit equation above if the voltage source is constant for all time. In other words, $V(t) = K$ for all t. (Your solution will contain the three parameters, R, C, and K, along with a constant that depends on the initial condition.)

20. Find the solution for the voltage $v_c(t)$ with initial value $v_c(0) = 1$ in the RC circuit equation given above if the voltage source $V(t)$ is the step function given by

$$V(t) = \begin{cases} K & \text{for } 0 \le t < 3; \\ 0 & \text{for } t > 3. \end{cases}$$

Your answer should contain the three parameters R, C, and K.

21. Given the source voltage $V(t) = 2t$ and the parameter values $R = 0.2$ and $C = 1$,

(a) sketch the slope field using `HPGSolver`,

(b) sketch the graph of the solution with the initial condition $v_c(0) = 0$ without using any technology,

(c) sketch the graph of the solution with the initial condition $v_c(0) = 3$ without using any technology, and

(d) confirm your answer using `HPGSolver`.

22. Given the source voltage

$$V(t) = \begin{cases} 0 & \text{for } 0 \le t < 1; \\ 2 & \text{for } t \ge 1; \end{cases}$$

and the parameter values $R = 0.2$ and $C = 1$,

(a) sketch the slope field using `HPGSolver`,

(b) sketch the graph of the solution with the initial condition $v_c(0) = 0$ without using any technology,

(c) sketch the graph of the solution with the initial condition $v_c(0) = 3$ without using any technology, and

(d) confirm your answer using `HPGSolver`.

23. Given the source voltage

$$V(t) = \begin{cases} 2t & \text{for } 0 \le t < 1; \\ 2 & \text{for } t \ge 1; \end{cases}$$

and parameter values $R = 0.2$ and $C = 1$,

(a) sketch the slope field using `HPGSolver`,

(b) sketch the graph of the solution with the initial condition $v_c(0) = 0$ without using any technology,

(c) sketch the graph of the solution with the initial condition $v_c(0) = 3$ without using any technology,

(d) confirm your answer using `HPGSolver`, and

(e) discuss in a few sentences the differences between the solutions for this differential equation and the solutions for the differential equations in Exercises 21 and 22.

24. Suppose that a population can be accurately modeled by the logistic equation

$$\frac{dp}{dt} = 0.4p\left(1 - \frac{p}{30}\right).$$

(Note that the growth-rate parameter is 0.4 and the carrying capacity is 30.) Suppose that, at time $t = 5$, a disease is introduced into the population that kills 25% of the population per year. To adjust the model, we change the differential equation to

$$\frac{dp}{dt} = \begin{cases} 0.4p\left(1 - \frac{p}{30}\right) & \text{for } 0 \leq t < 5; \\ 0.4p\left(1 - \frac{p}{30}\right) - 0.25p & \text{for } t > 5. \end{cases}$$

(a) Sketch the slope field for this equation using ~~HPGSolver~~.

(b) Using the slope field, sketch the graphs of a few representative solutions to this equation.

(c) Find formulas for the solutions of this equation for initial conditions $p(0) = 30$ and $p(0) = 20$.

(d) In a few sentences, describe the behavior of the solutions with initial conditions $p(0) = 30$ and $p(0) = 20$. (You can use either the sketches from the slope field or the formulas, but give a qualitative description of the solutions.)

1.4 NUMERICAL TECHNIQUE: EULER'S METHOD

The geometric concept of a slope field as discussed in the previous section is closely related to a fundamental numerical method for approximating solutions to a differential equation. Given an initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

we can get a rough idea of the graph of its solution by first sketching the slope field in the ty-plane and then, starting at the initial value (t_0, y_0), sketching the solution by drawing a graph that is tangent to the slope field at each point along the graph. In this section we describe a numerical procedure that automates this idea. Using a computer or a calculator, we obtain numbers and graphs that approximate solutions to initial-value problems.

Numerical methods provide quantitative information about solutions even if we cannot find their formulas. There is also the advantage that most of the work can be done by machine. The disadvantage is that we obtain only approximations, not precise solutions. If we remain aware of this fact and are prudent, numerical methods become powerful tools for the study of differential equations. It is not uncommon to turn to numerical methods even when it is possible to find formulas for solutions. (Most of the graphs of solutions of differential equations in this text were drawn using numerical approximations even when formulas were available.)

The numerical technique that we discuss in this section is called *Euler's method.* A more detailed discussion of the accuracy of Euler's method as well as other numerical methods is given in Chapter 7.