

Rule of thumb for second guessing

The last example indicates what is so unsatisfying about guessing techniques. How did we know to make the second guess a product of t and our first guess? The answer is that we have either seen a similar problem before or we can figure out at least the form of the guess by another technique. Methods for arriving at the second guess with less guesswork but more computation are given in Exercise 23 of Section 1.9, in Chapter 6, and in Exercises 17 and 18 of Appendix B.

EXERCISES FOR SECTION 1.8

In Exercises 1–6, find the general solution of the equation specified.

1. $\frac{dy}{dt} = y + 3e^{-2t}$

2. $\frac{dy}{dt} = -4y + 3e^{-t}$

3. $\frac{dy}{dt} = y + \cos 2t$

4. $\frac{dy}{dt} = 2y + \sin 2t$

5. $\frac{dy}{dt} = 4y - 5e^{4t}$

6. $\frac{dy}{dt} = \frac{y}{2} + 4e^{t/2}$

In Exercises 7–12, solve the given initial-value problem.

7. $\frac{dy}{dt} + 2y = e^{t/3}, \quad y(0) = 1$

8. $\frac{dy}{dt} - 2y = 3e^{-2t}, \quad y(0) = 10$

9. $\frac{dy}{dt} + y = \cos 2t, \quad y(0) = 5$

10. $\frac{dy}{dt} + 3y = \cos 2t, \quad y(0) = -1$

11. $\frac{dy}{dt} + 5y = 3e^{-5t}, \quad y(0) = -2$

12. $\frac{dy}{dt} - 2y = 7e^{2t}, \quad y(0) = 3$

13. Consider the nonhomogeneous linear equation

$$\frac{dy}{dt} + 2y = \cos 3t.$$

To find a particular solution, it is pretty clear that our guess must contain a cosine function, but it is not so clear that the guess must also contain a sine function.

(a) Guess $y_p(t) = \alpha \cos 3t$ and substitute this guess into the equation. Is there a value of α such that $y_p(t)$ is a solution?

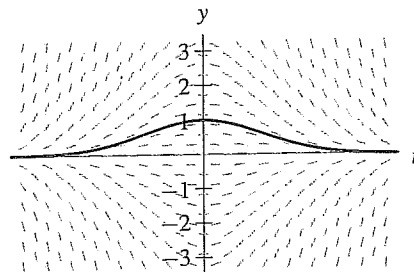
(b) Write a brief paragraph explaining why the proper guess for a particular solution is $y_p(t) = \alpha \cos 3t + \beta \sin 3t$.

14. Consider the nonhomogeneous linear equation

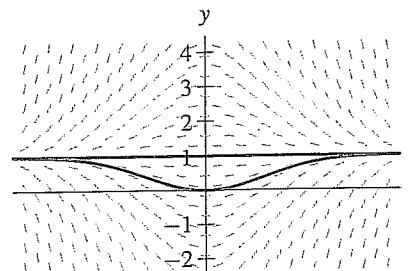
$$\frac{dy}{dt} = \lambda y + \cos 2t.$$

To find its general solution, we add the general solution of the associated homogeneous equation and a particular solution $y_p(t)$ of the nonhomogeneous equation. Briefly explain why it does not matter which solution of the nonhomogeneous equation we use for $y_p(t)$.

15. The graph to the right is the graph of a solution of a homogeneous linear equation $dy/dt = a(t)y$. Give rough sketches of the graphs of the solutions to this equation that satisfy the initial conditions $y(0) = 0$, $y(0) = 2$, $y(0) = 3$, $y(0) = -1$, and $y(0) = -2.5$.



16. The two graphs to the right are graphs of solutions of a nonhomogeneous linear equation $dy/dt = a(t)y + b(t)$. Give rough sketches of the graphs of the solutions to this equation that satisfy the initial conditions $y(0) = 2$, $y(0) = 3.5$, $y(0) = -1$, and $y(0) = -2$.



17. Consider the nonlinear differential equation $dy/dt = y^2$.
- Show that $y_1(t) = 1/(1-t)$ is a solution.
 - Show that $y_2(t) = 2/(1-t)$ is not a solution.
 - Why don't these two facts contradict the Linearity Principle?
18. Consider the nonhomogeneous linear equation $dy/dt = -y + 2$.
- Compute an equilibrium solution for this equation.
 - Verify that $y(t) = 2 - e^{-t}$ is a solution for this equation.
 - Using your results in parts (a) and (b) and the Uniqueness Theorem, explain why the Linearity Principle does not hold for this equation.
19. Consider a nonhomogeneous linear equation of the form

$$\frac{dy}{dt} + a(t)y = b_1(t) + b_2(t),$$

that is, $b(t)$ is written as a sum of two functions. Suppose that $y_h(t)$ is a solution of the associated homogeneous equation $dy/dt + a(t)y = 0$, that $y_1(t)$ is a solution of the equation $dy/dt + a(t)y = b_1(t)$, and that $y_2(t)$ is a solution of the equation $dy/dt + a(t)y = b_2(t)$. Show that $y_h(t) + y_1(t) + y_2(t)$ is a solution of the original nonhomogeneous equation.

20. Consider the nonhomogeneous linear equation

$$\frac{dy}{dt} + 2y = 3t^2 + 2t - 1.$$

In order to find the general solution, we must guess a particular solution $y_p(t)$. Since the right-hand side is a quadratic polynomial, it is reasonable to guess a quadratic for $y_p(t)$, so let

$$y_p(t) = at^2 + bt + c,$$

where a , b , and c are constants. Determine values for these constants so that $y_p(t)$ is a solution.

In Exercises 21–24, find the general solution and the solution that satisfies the initial condition $y(0) = 0$.

21. $\frac{dy}{dt} + 2y = t^2 + 2t + 1 + e^{4t}$

22. $\frac{dy}{dt} + y = t^3 + \sin 3t$

23. $\frac{dy}{dt} + 3y = \cos 2t + e^{3t} + e^{-4t}$

24. $\frac{dy}{dt} + y = \cos 2t + 3 \sin 2t + e^{-t}$

In Exercises 25–28, give a brief qualitative description of the behavior of solutions. Note that we only give partial information about the functions in the differential equation, so your description must allow for various possibilities. Be sure to deal with initial conditions of different sizes and to discuss the long-term behavior of solutions.

25. $\frac{dy}{dt} + 2y = b(t)$, where $-1 < b(t) < 2$ for all t .

26. $\frac{dy}{dt} - 2y = b(t)$, where $-1 < b(t) < 2$ for all t .

27. $\frac{dy}{dt} + y = b(t)$, where $b(t) \rightarrow 3$ as $t \rightarrow \infty$.

28. $\frac{dy}{dt} + ay = \cos 3t + b$, where a and b are positive constants.

29. A person initially places \$500 in a savings account that pays interest at the rate of 4% per year compounded continuously. Suppose the person arranges for \$10 per week to be deposited automatically into the savings account.

(a) Write a differential equation for $P(t)$, the amount on deposit after t years (assume that “weekly deposits” is close enough to “continuous deposits” so that we may model the balance with a differential equation.)

(b) Find the amount on deposit after 5 years.

EXERCISES FOR SECTION 1.9

In Exercises 1–6, find the general solution of the differential equation specified.

1. $\frac{dy}{dt} = -\frac{y}{t} + 2$

2. $\frac{dy}{dt} = \frac{3}{t}y + t^5$

3. $\frac{dy}{dt} = -\frac{y}{1+t} + t^2$

4. $\frac{dy}{dt} = -2ty + 4e^{-t^2}$

5. $\frac{dy}{dt} - \frac{2t}{1+t^2}y = 3$

6. $\frac{dy}{dt} - \frac{2}{t}y = t^3e^t$

In Exercises 7–12, solve the given initial-value problem.

7. $\frac{dy}{dt} = -\frac{y}{1+t} + 2, \quad y(0) = 3$

8. $\frac{dy}{dt} = \frac{1}{t+1}y + 4t^2 + 4t, \quad y(1) = 10$

9. $\frac{dy}{dt} = -\frac{y}{t} + 2, \quad y(1) = 3$

10. $\frac{dy}{dt} = -2ty + 4e^{-t^2}, \quad y(0) = 3$

11. $\frac{dy}{dt} - \frac{2y}{t} = 2t^2, \quad y(-2) = 4$

12. $\frac{dy}{dt} - \frac{3}{t}y = 2t^3e^{2t}, \quad y(1) = 0$

In Exercises 13–18, the differential equation is linear, and in theory, we can find its general solution using the method of integrating factors. However, since this method involves computing two integrals, in practice it is frequently impossible to reach a formula for the solution that is free of integrals. For these exercises, determine the general solution to the equation and express it with as few integrals as possible.

13. $\frac{dy}{dt} = (\sin t)y + 4$

14. $\frac{dy}{dt} = t^2y + 4$

15. $\frac{dy}{dt} = \frac{y}{t^2} + 4 \cos t$

16. $\frac{dy}{dt} = y + 4 \cos t^2$

17. $\frac{dy}{dt} = -\frac{y}{e^{t^2}} + \cos t$

18. $\frac{dy}{dt} = \frac{y}{\sqrt{t^3-3}} + t$

19. For what value(s) of the parameter a is it possible to find explicit formulas (without integrals) for the solutions to

$$\frac{dy}{dt} = aty + 4e^{-t^2}?$$

20. For what value(s) of the parameter r is it possible to find explicit formulas (without integrals) for the solutions to

$$\frac{dy}{dt} = t^r y + 4?$$

21. Consider the nonhomogeneous equation

$$\frac{dv}{dt} + 0.4v = 3 \cos 2t.$$

- (a) Find the general solution using the method of integrating factors.
 (b) Find the general solution using the guessing technique from Section 1.8.

Comment on which method was easier for you.

22. In this exercise, we explore the connections between the method of integrating factors discussed in this section and the Extended Linearity Principle. Consider the nonhomogeneous linear equation

$$\frac{dy}{dt} = a(t)y + b(t),$$

where $a(t)$ and $b(t)$ are continuous for all t .

(a) Let

$$\mu(t) = e^{-\int_0^t a(\tau) d\tau}.$$

Show that $\mu(t)$ is an integrating factor for the nonhomogeneous equation.

- (b) Show that $1/\mu(t)$ is a solution to the associated homogeneous equation.
 (c) Show that

$$y_p(t) = \frac{1}{\mu(t)} \int_0^t \mu(\tau) b(\tau) d\tau$$

is a solution to the nonhomogeneous equation.

- (d) Use the Extended Linearity Principle to find the general solution of the nonhomogeneous equation.
 (e) Compare your result in part (d) to the formula

$$y(t) = \frac{1}{\mu(t)} \int \mu(t) b(t) dt$$

for the general solution that we obtained on page 128.

- (f) Illustrate the calculations that you did in this exercise for the example

$$\frac{dy}{dt} = -2ty + 4e^{-t^2}.$$

23. Consider the nonhomogeneous equation

$$\frac{dy}{dt} + 2y = 3e^{-2t}.$$

In Section 1.8, we saw that the guess $y_p(t) = \alpha e^{-2t}$ does not produce a solution because it is a solution to the associated homogeneous equation. We then guessed $y_p(t) = \alpha t e^{-2t}$. Use the method of integrating factors to explain why this guess is a good idea.