42. Suppose you are having a dinner party for a large group of people, and you decide to make 2 gallons of chili. The recipe calls for 2 teaspoons of hot sauce per gallon, but you misread the instructions and put in 2 tablespoons of hot sauce per gallon. (Since each tablespoon is 3 teaspoons, you have put in 6 teaspoons per gallon, which is a total of 12 teaspoons of hot sauce in the chili.) You don't want to throw the chili out because there isn't much else to eat (and some people like hot chili), so you serve the chili anyway. However, as each person takes some chili, you fill up the pot with beans and tomatoes without hot sauce until the concentration of hot sauce agrees with the recipe. Suppose the guests take 1 cup of chili per minute from the pot (there are 16 cups in a gallon), how long will it take to get the chili back to the recipe's concentration of hot sauce? How many cups of chili will have been taken from the pot?

43. In Exercise 12 of Section 1.1, we saw that the velocity v of a freefalling skydiver is well modeled by the differential equation

$$m\frac{dv}{dt} = mg - kv^2,$$

where m is the mass of the skydiver, g is the gravitational constant, and k is the drag coefficient determined by the position of the driver during the dive.

(a) Find the general solution of this differential equation.

(b) Confirm your answer to Exercise 12 of Section 1.1 by calculating the limit of $v(t)$ as $t \to \infty$.

1.3 QUALITATIVE TECHNIQUE: SLOPE FIELDS

Finding an analytic expression (in other words, finding a formula) for a solution to a differential equation is often a useful way to describe a solution of a differential equation. However, there are other ways to describe solutions, and these alternative representations are frequently easier to understand and use. In this section we focus on geometric techniques for representing solutions, and we develop a method for visualizing the graphs of the solutions to the differential equation

$$\frac{dy}{dt} = f(t, y).$$

The Geometry of $dy/dt = f(t, y)$

If the function $y(t)$ is a solution of the equation $dy/dt = f(t, y)$ and if its graph passes through the point (t_1, y_1) where $y_1 = y(t_1)$, then the differential equation says that the derivative dy/dt at $t = t_1$ is given by the number $f(t_1, y_1)$. Geometrically, this equality of dy/dt at $t = t_1$ with $f(t_1, y_1)$ means that the slope of the tangent line to the graph of $y(t)$ at the point (t_1, y_1) is $f(t_1, y_1)$ (see Figure 1.10). Note that there is nothing special about the point (t_1, y_1) other than the fact that it is a point on the graph

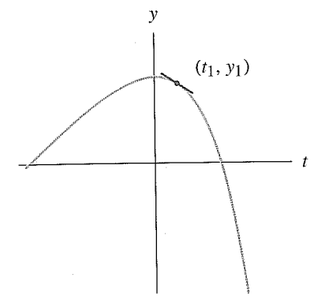

Figure 1.10
Slope of the tangent at the point (t_1, y_1) is given by the value of $f(t_1, y_1)$.

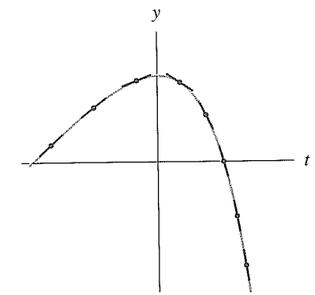

Figure 1.11
If $y = y(t)$ is a solution, then the slope of any tangent must equal $f(t, y)$.

of the solution $y(t)$. The equality of dy/dt and $f(t, y)$ must hold for all t for which $y(t)$ satisfies the differential equation. In other words, the values of the right-hand side of the differential equation yield the slopes of the tangents at all points on the graph of $y(t)$ (see Figure 1.11).

Slope Fields

This simple geometric observation leads to our main device for the visualization of the solutions to a first-order differential equation

$$\frac{dy}{dt} = f(t, y).$$

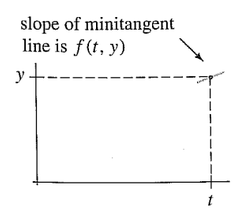

Figure 1.12
The slope of the minitangent at the point (t, y) is determined by the right-hand side $f(t, y)$ of the differential equation.

If we are given the function $f(t, y)$, we obtain a rough idea of the graphs of the solutions to the differential equation by sketching its corresponding **slope field**. We make this sketch by selecting points in the ty-plane and computing the numbers $f(t, y)$ at these points. At each point (t, y) selected, we use $f(t, y)$ to draw a minitangent line whose slope is $f(t, y)$ (see Figure 1.12). These minitangent lines are also called slope marks. Once we have a lot of slope marks, we can visualize the graphs of the solutions. For example, consider the differential equation

$$\frac{dy}{dt} = y - t.$$

In other words, the right-hand side of the differential equation is given by the function $f(t, y) = y - t$. To get some practice with the idea of a slope field, we sketch its slope field by hand at a small number of points. Then we discuss a computer-generated version of this slope field.

Generating slope fields by hand is tedious, so we consider only the nine points in the ty-plane. For example, at the point $(t, y) = (1, -1)$, we have $f(t, y) = f(1, -1) = -1 - 1 = -2$. Therefore we sketch a "small" line segment with slope -2 centered at