M224

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Analytic methods: separation

General 1^{st} order DE: $\boxed{y' = f(y,t)}$

not solvable analytically !!

Special classes: 1° $y' = f(y)\,g(t)$ – separable

2° $y' = f(y)$ – autonomous

Examples:

1) Linear growth-decay: $y' = ay$

$y' = ay + b$

2) Logistic: $y' = ay(1 - y/N) \pm b$

3) Chemical reactions:
$$\begin{cases} y' = k_2 (a-y)(b-y) \\ y' = k_2 (a-y)^2 \\ \dots \end{cases}$$

4) Geometric:

$y' = \sqrt{\left(\frac{y}{a}\right)^2 - 1}$ – heavy chain; Min surf revolut.

$y' = \sqrt{\frac{a+y}{H-y}}$ – fastest slope

5) Kepler 2-body: $\frac{dr}{d\theta} = \underbrace{\frac{r}{J}\sqrt{2(Er^2 + GMr) - J^2}}_{f(r)}$

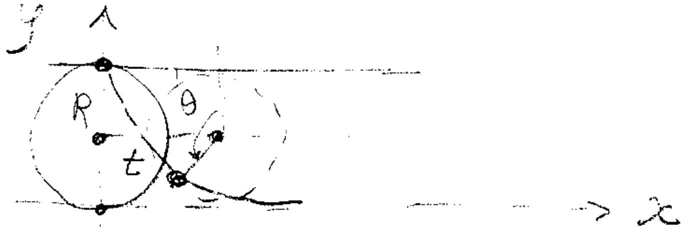
Separation: $\boxed{\frac{dy}{dt} = f(y)\,g(t)} \Rightarrow \underbrace{\int \frac{dy}{f(y)}}_{F(y)} = C + \underbrace{\int g(t)\,dt}_{G(t)}$

Implicit: $\boxed{F(y) = C + G(t)}$

Explicit: $\boxed{y(t,c) = F^{-1}(c + G(t))}$ - gen. solut.

DE	Integral	Implicit	Explicit	IVP
$y' = ay$	$\int \frac{dy}{y} = c_1 + a\int dt$	$\ln y = c_1 + at$	$y = Ce^{at}$	$y = y_0 e^{at}$
$y' = ay + b$	$\int \frac{dy}{y + b/a} = c_1 + \int a\,dt$	$\ln(y + b/a) = c_1 + at$	$y = Ce^{at} - b/a$	$y = (y_0 + \frac{b}{a})e^{at} - b/a$
$y' = ay(1 - y/N)$	$\int \frac{dy}{y(1 - y/N)} = c_1 + at$	$\ln \frac{y}{N-y} = c_1 + at$	$\frac{N}{1 - Ce^{-at}}$	$\frac{Ny_0}{y_0 + (N - y_0)e^{-at}}$
$y' = (a-y)(b-y)$, $a \neq b$	$\int \frac{dy}{(a-y)(b-y)} = c_1 + t$	$\frac{1}{a-b} \ln \frac{a-y}{b-y} = c_1 + t$	$\frac{a - Cbe^{(a-b)t}}{1 - Ce^{(a-b)t}}$	- - -
$y' = (a-y)^2$	$\int \frac{dy}{(a-y)^2} = c + t$	$\frac{1}{a-y} = c + t$	$y = a - \frac{1}{c+t}$	$y = a - \frac{1}{t + 1/(a-y_0}$
$y' = \sqrt{y^2 - a^2}$	$\int \frac{dy}{\sqrt{y^2 - a^2}} = c_1 + t$	$\ln(y + \sqrt{y^2 - a^2}) = c_1 + t$	$a \cosh(t + c)$	$a\,ch(t + ch^{-1}(\frac{y}{a}$
$y' = \sqrt{\frac{y}{H-y}}$	$\int \sqrt{}\, dy = c_1 + t$	$\sqrt{y(H-y)} + H \tan^{-1}\sqrt{y/(H-y)} = c_1 + t$	cycloid	

Cycloid:



Center: (t, R)

Angle: $\theta = t/R$

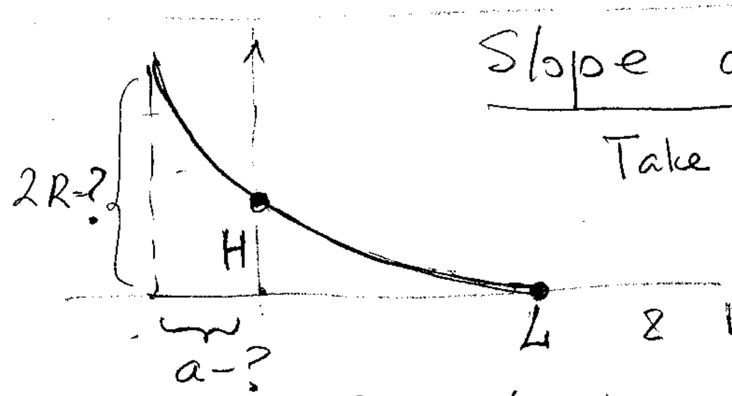
$$\begin{cases} x = t - R\sin t/R \\ y = R(\underbrace{1 + \cos t/R}_{2\cos^2\theta/2}) \end{cases} \Rightarrow \begin{cases} \cos\theta/2 = \sqrt{y/2R} \\ \sin\theta/2 = \sqrt{1 - y/2R} \\ \tan\theta/2 = \sqrt{\dfrac{2R-y}{y}} \end{cases}$$

$$\Rightarrow \quad x = 2R\cos^{-1}\sqrt{\frac{y}{2R}} - 2R\sqrt{\frac{y}{2R}\left(1 - y/2R\right)}$$

$$\boxed{x = -\sqrt{y(2R-y)} + 2R\tan^{-1}\sqrt{\frac{2R-y}{y}}}$$

Also check DE for cycloid

$$\begin{cases} dx = (1 - \cos t/R)\,dt \\ dy = -\sin t/R \quad dt \end{cases} \Rightarrow \boxed{\frac{dy}{dx} = -\cot\theta/2 = \sqrt{\frac{y}{2R-y}}}$$



Slope design: need a - ? R - ?

Take solution:

$$x + a = 2R\tan^{-1}\sqrt{\frac{2R-y}{y}} - \sqrt{y(2R-y)}$$

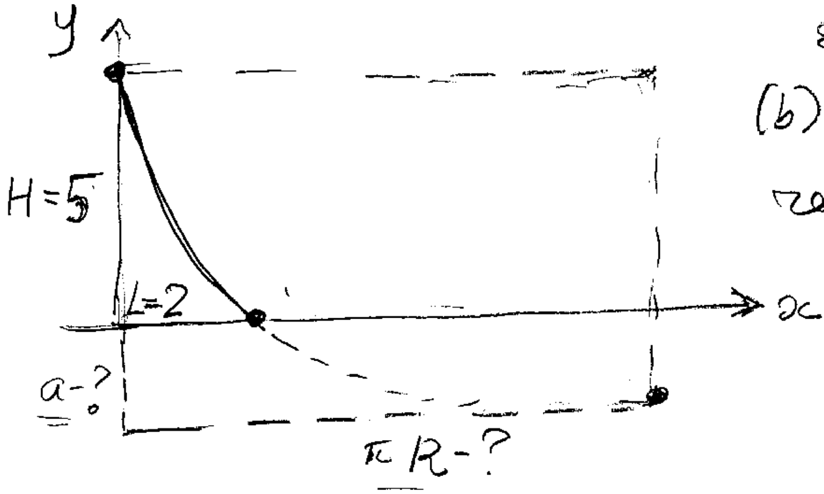
& Use boundary conditions (end-points)

$$\begin{array}{ll} (x,y) = (0,H): & \\ (x,y) = (L,0): & \end{array} \begin{cases} a = 2R\tan^{-1}\sqrt{\dfrac{2R-H}{H}} - \sqrt{H(2R-H)} = 0 \\ L + a = \pi R \end{cases}$$

Eq-n for radius R

$$\boxed{\pi R - L = 2R\tan^{-1}\sqrt{\frac{2R-H}{H}} - \sqrt{H(2R-H)}} \quad ?$$

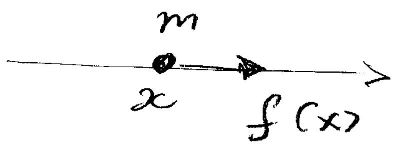
Problem: Design fastest slope for $H=5$; $L=2$ (Hint: use y-offset a instead of x-offset). (a) Find a, R, and plot solution.



(b) What condition on H, L requires x-offset, or y-offset, or no offset.

Particle in potential force

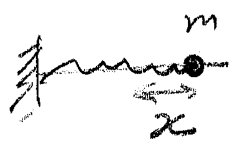
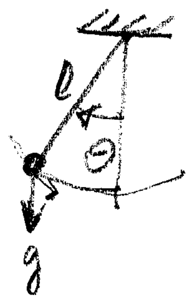
$$f(x) = -U'(x)$$



$$\boxed{m\ddot{x} = -U'(x)}$$

2nd order Newton DE for position $x(t)$

Examples:

	$f(x)$	$U(x)$
Oscillator: 	$-kx$	$\frac{kx^2}{2}$
Pendulum: 	$-g\sin\theta$	$-g\cos\theta$

Energy conservation:

$$E = \frac{m\dot{x}^2}{2} + U(x) \quad \text{— const}$$

Solve for $\dot{x}$

$$\boxed{\dot{x} = \pm\sqrt{\frac{2}{m}\left[E - U(x)\right]}}$$

1st order separable DE for $x(t)$

Linear oscillator:

$$\dot{x} = \sqrt{\frac{2}{m}\left(E - \frac{kx^2}{2}\right)} \Rightarrow \text{separate}$$

$$\int \frac{dx}{\sqrt{\frac{2}{m}\left(E - \frac{kx^2}{2}\right)}} = t + c_1$$

Integrate: $\int \frac{dx}{\sqrt{a - bx^2}} = \frac{1}{\sqrt{b}} \sin^{-1}\left(\sqrt{\frac{b}{a}}\, x\right)$

Solve for x:

$$x(t) = \sqrt{\frac{a}{b}} \sin \sqrt{b}\,(t + c_1)$$

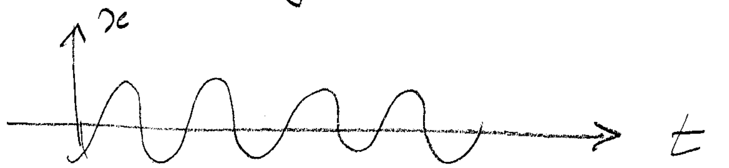
For oscillator: $a = \frac{2E}{m}$; $b = \frac{k}{m}$

Get oscillating solution:

$$\boxed{x(t) = A \sin\omega (t + c_1)}$$

of amplitude: $A = \sqrt{\frac{2E}{m}}$

frequency: $\omega = \sqrt{k/m}$



Ex: $m = 1$; $k = 4$ $\Rightarrow$ $\boxed{\ddot{x} = -4x}$ $\Rightarrow$ $\boxed{x(t) = A\cos 2t}$