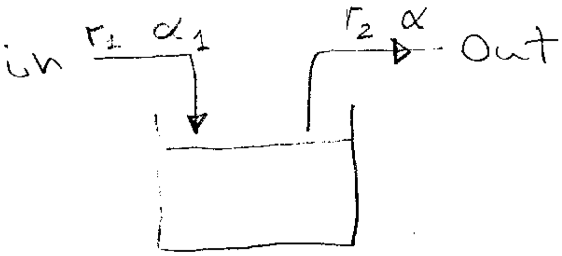
Mixing problems



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$$\left\{\begin{array}{l} r = \text{rate}\left[\frac{cm^3}{sec}\right] \\ \alpha = \text{concent}\left[gr/cm^3\right] \\ q = [gr] \\ V = \text{volume}\left[cm^3\right] \\ q = V\alpha \text{ - quantity} \end{array}\right.$$

Balance:

1) $\Delta V = (r_1 - r_2)\Delta t \Rightarrow \boxed{V' = (r_1 - r_2)}$

2) $\Delta q = \left(\alpha_1 r_1 - \frac{q}{V} r_2\right)\Delta t$

$$\Rightarrow \boxed{q' + \frac{r_2}{V} q = \alpha_1 r_1}$$

1° Equal rates $\boxed{r_1 = r_2} \Rightarrow V = V_0$ - const

$$\Rightarrow q(t) = \underbrace{\frac{\alpha_1 r_1}{r_2/V}}_{\alpha_1 V} + \left(q_0 - \underbrace{\frac{\alpha_1 r_1}{r_2/V}}_{\alpha_1 V}\right)e^{-\frac{r_2}{V}t}$$

2° Zero incoming conc. $\boxed{q' + \frac{r_2}{V} q = 0}$

$\alpha_1 = 0$

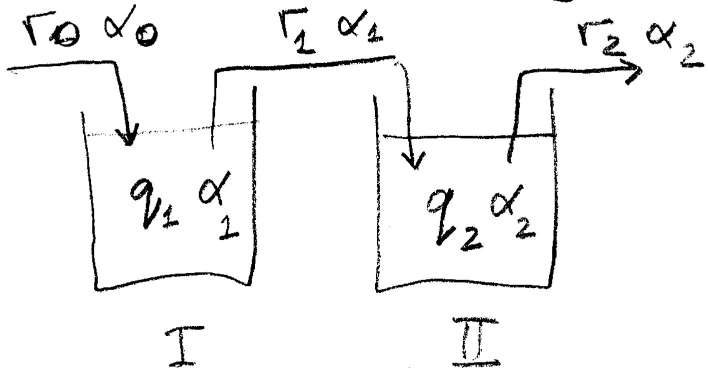
$$\Rightarrow \boxed{q(t) = q_0 e^{-\int \frac{r_2}{V} dt}}$$

3° Different rates:

$V = V_0 + (r_1 - r_2)t$

$$\boxed{q' + \frac{r_2}{V} q = \alpha_1 r_1} \leftarrow \text{Non separable (Linear DE)}$$

Problem 2: Write differential eq-n(s) model for a system of two tanks:



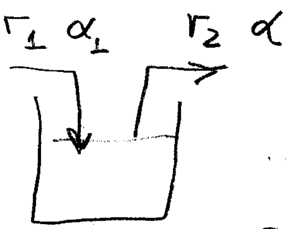
$\left\{\begin{array}{l} r_0 \text{ - incoming} \\ r_1 \text{ - } (I \to II) \\ r_2 \text{ - outgoing} \end{array}\right\}$ rates

$\left\{\begin{array}{l} \alpha_0 \text{ - incoming conc.} \\ \alpha_1 \text{ - tank I} \\ \alpha_2 \text{ - tank II} \end{array}\right.$

$\left\{\begin{array}{l} q_1 \\ q_2 \end{array}\right\}$ amounts

$$\begin{cases} \dfrac{dq_1}{dt} = \dots \\[2ex] \dfrac{dq_2}{dt} = \dots \end{cases}$$

Problem 3:

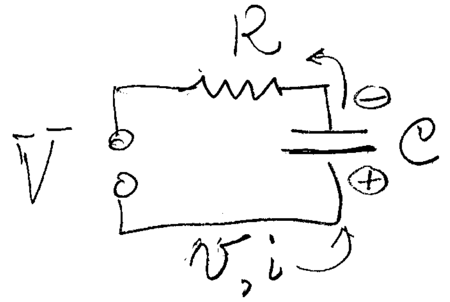


Tank is filled at rate $r_1 = 1\ m^3/s$ and emptied at rate $r_2 = 2.5\ m^3/s$, with fresh water ($\alpha_1 = 0$).

The initial volume $V_0 = 50 m^3$ and concentr. $\alpha_0 = 20\ \frac{mg}{m^3}$. (i) Find & plot concentration $\alpha(t)$

(ii) Compute its value when tank is half empty and $\alpha(t_{fin})$ when volume $= 0$.

CR circuits (Kirchoff)



i – current (flow of charge)
v – voltage (potential force driving charge)
q – charge (accumulating at C)

Kirchoff laws:

(i) $\boxed{v = R i}$ (Ohm) – voltage applied to R creates current proportional to v/R. R – resistance.

(ii) $\boxed{q = C v}$ – charge accumulation creates voltage proportional to q/C. C – capacitance.

(iii) $\boxed{\frac{dq}{dt} = -i}$ – current is due to "moving charges" on C

$$C \frac{dv}{dt} = -\frac{v}{R} \Rightarrow \boxed{v' = -\frac{1}{RC} v + V}$$

V ← voltage source

Hydrodynamic analogy

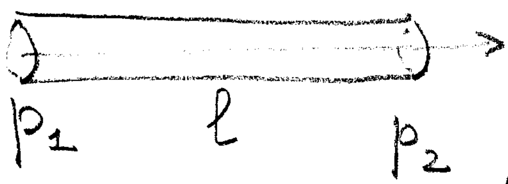
* voltage = pressure drop ($\propto$ height h in tank)
* current = velocity U (or volume flux F)
* charge = mass/volume

(i) current → $F = \frac{\pi a^4}{8\nu} \frac{g}{\ell} h$ ← voltage; $\frac{\pi a^4}{8\nu} \frac{g}{\ell}$ ← $1/R$

(ii) mass/vol. $= A h$; A ← capacitance

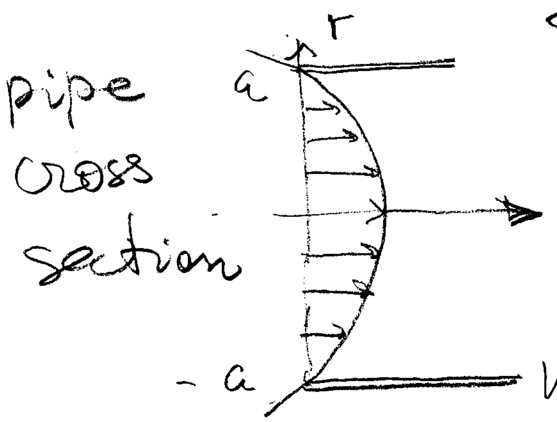
Poiseuille pipe-flow & CR circuits

① Poiseuille — a stationary fluid flow



in pipe of radius a (cross section πa^2) created by pressure gradient $\frac{\Delta p}{\ell} = \frac{p_1 - p_2}{\ell}$

has velocity profile:



$$U(r) = \frac{\Delta p}{\ell} \frac{a^2 - r^2}{4\nu};$$

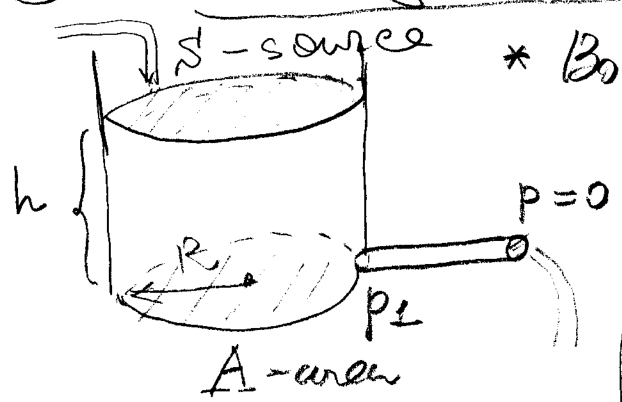
ν – viscosity ($8 \cdot 10^{-7} \frac{m^2}{s}$ for water)

Volume (mass) flux: $F = \iint \rho U \, dA$

$$F = \boxed{\frac{\pi a^4}{8} \frac{\Delta p}{\ell \nu}} \rho;$$

ρ – mass density

② Leaking tank:



* Bottom (hydrostatic) pressure:

$$\boxed{p_1 = g h; \quad p_2 = 0}$$

Volume loss: $\frac{dV}{dt} = F$

$$\boxed{A \frac{dh}{dt} = -\frac{\pi a^4 g}{8 \ell \nu} h + S}$$

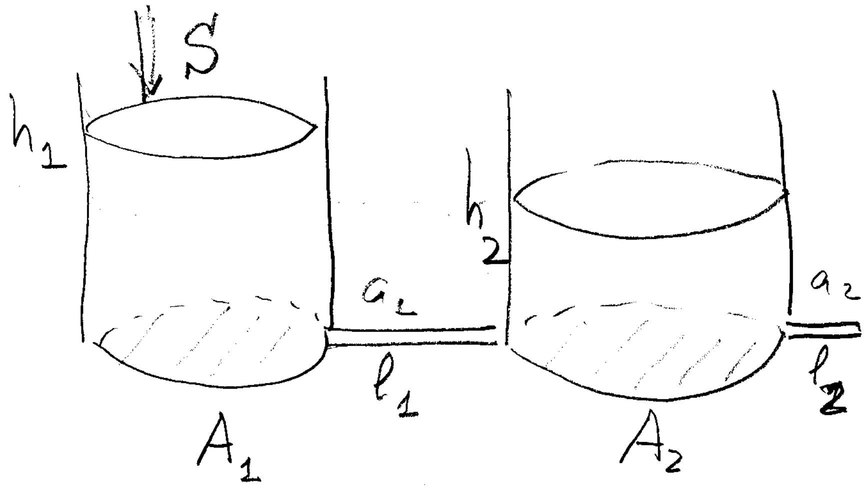
Linear 1st order DE: $\boxed{h' = -a h + b}$

Problem: (i) How long does it take to drain half tank of water (kinematic viscosity $\nu = 8 \cdot 10^{-7}\ m^2/s$) filled at height $h_0 = 5\,m$ through a pipe of radius $a = 2\,cm$, length $l = 2\,m$

(ii) Can you empty the tank?

(iii) The water is supplied to the tank at a rate $S = .3\ m^3/s$ and drained through the same pipe. What equilibrium level h it will attain?

Problem 2: Set up a DE model for a 2-tank system connected by pipes of radii (a_1, a_2), length l_1, l_2, and source "S" (4)



$$\begin{cases} \dfrac{dh_1}{dt} = \dots \\ \dfrac{dh_2}{dt} = \dots \end{cases}$$