Applications of LDE

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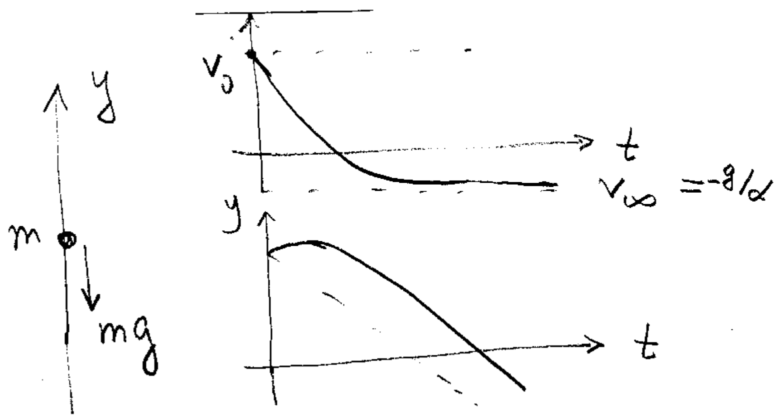
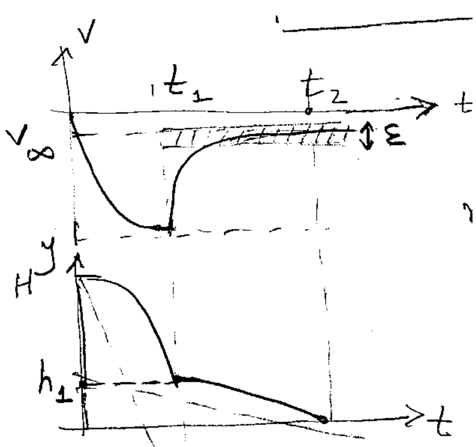
I. Free fall with friction

y(t) - position (height); $v(t)=\dot{y}(t)$ - velocity.

Newton DE: $$\begin{cases} \dot{v}=-\alpha v-g & \rightarrow \quad v(t)=-g/\alpha+(v_0+g/\alpha)e^{-\alpha t} \\ \dot{y}=v & \rightarrow \quad y(t)=y_0-\dfrac{g}{\alpha}t+(v_0+g/\alpha)\dfrac{1-e^{-\alpha t}}{\alpha} \end{cases}$$



Parachute jump (2-stage fall)



Goal: by the time of landing t want 'safe' limiting velocity v_∞

224: Basic demographics and decay with source

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Basic demographics

Life span L=1/d (death rate), f = fertility/female. Total population y(t) obeys a LDE (linear DE):

(1) $y' = ay$; with P.C.GR (coefficient) $a = \left(\frac{f}{2} - 1\right)/L$.

Problem 1: i) Explain (derive) the basic demographic LDE-model in terms of birth/death rates

ii) Augment equation (1) with emigration term proportional to y(t) and P.C. migration rate = m

iii) find *replacement* fertility f (to sustain a steady population) in terms of (L,m)

iv) Take f=2.3, L=60 years, m=.03/year. Show that y(t) decreases with time, estimate how long it will take for population y(t) to drop to 1/3.

Decay with source (parameter fit)

First order decay reaction for concentration y(t) with source b: $b \to y \overset{k}{\to} \ldots$, obeys DE:
$y' = -ky + b; y(0) = y_0$. We use IVP solution $y(t) = q + (y_0 - q)e^{-kt}$, with unknown equilibrium q,

and the data (from a decay experiment)

t	0	Δt	$2\Delta t$
y	y_0	y_1	y_2

, to estimate parameters {*k,b*} in terms of

differences: $\Delta t; \Delta y_0 = y_1 - y_0; \Delta y_1 = y_2 - y_1$

$$\left.\begin{aligned} y_1 - q &= (y_0 - q)e^{-k\Delta t} \\ y_2 - q &= (y_1 - q)e^{-k\Delta t} \end{aligned}\right\} \Rightarrow \begin{aligned} w = e^{-k\Delta t} &= \frac{\Delta y_1}{\Delta y_0} \Rightarrow k = \ldots \\ q &= \frac{y_1 - w y_0}{1 - w} \Rightarrow b = \ldots \end{aligned}$$

Problem 2: Apply these formulae to the following data:

t	0	3	6
y	1.2	$.5$	$.3$

; estimate (k,b) and plot solution (vs. data)

II. Financing loan

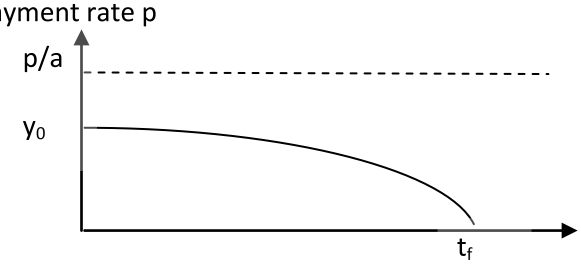
1) y_0 - initial loan (principal); α - annual interest rate (simple or compounded)

debt after t years:

$$\begin{cases} \text{annual: } y_0\left(1+\alpha\right)^t \\ \text{monthly: } y_0\left(1+\alpha/12\right)^{12t} \Longrightarrow y\left(t\right)\approx y_0e^{\alpha t} \\ \text{dayly: } y_0\left(1+\alpha/365\right)^{365t} \end{cases}$$

2) Continuous DE with interest α and payment rate p

IVP: $y' = \alpha y - p; y\left(0\right) = y_0$



3) solution: $y\left(t\right)=\frac{p}{\alpha}-\left(\frac{p}{\alpha}-y_0\right)e^{\alpha t}$;

4) Analysis: Minimal payment: $p_m = \alpha y_0$. To pay off need: $p_m/\alpha > y_0$ <==> $p > p_m$

Pay-off time: $t_f = \frac{1}{\alpha}\ln\left(\frac{p}{p-\alpha y_0}\right)$.

Total pay: $Y = p\bullet t_f = \frac{p}{\alpha}\ln\left(\frac{p}{p-\alpha y_0}\right) = y_0\bullet\frac{p}{p_m}\ln\left(\frac{p/p_m}{p/p_m-1}\right) = y_0 f\left(z\right); z = \frac{p}{p_m} > 1$

5) Questions: (i) Given {y_0 , a, p} find t; (ii) For fixed {a,y_0 ,t_f }, determine rate p

Problem 1: for α=7%; y_0 =200 => p_m =14; complete Table

Payment rate $p>p_m$	Fraction $z=p/p_m$	Total overpay: Y/y_0	Pay-off time t_f
21			
28			
40			

III. Exchange-Mixing systems

1. Newton cooling/heating

2. Transport/mixing

3. Migration (populations et al)

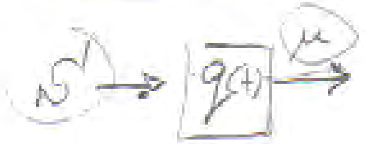
Mixing-transport systems

q - amount [mg] | r - flux [L/min]

V - volume [L] | S' - source [mg/min]

$\alpha = q/V$ - concentr. [mg/L] | μ - decay [1/min]

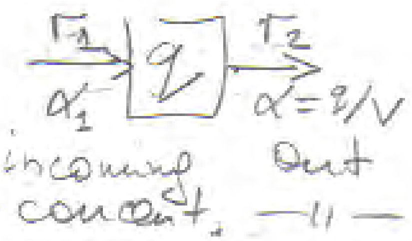
1. Source - decay



$$\dot q = S' - \mu q \iff \dot\alpha = \frac{S'}{V} - \mu\alpha \quad \text{(const } V\text{)}$$

Also for $V(t) \Rightarrow \frac{d}{dt}(\alpha V) = \ldots \Rightarrow \dot\alpha = \ldots$

2. In/out flux



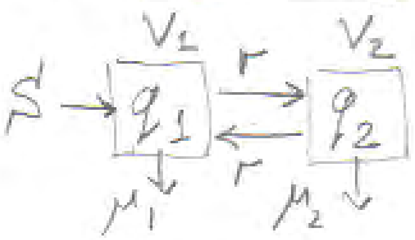
$$\dot V = (r_1 - r_2)$$

$$\dot q = \alpha_1 r_1 - r_2 \frac{q}{V}$$

→ 1° $(r_1 = r_2)$ V - const

→ 2° $(r_1 \neq r_2)$ $V(t)$

3. Exchange



(DS)
$$\begin{cases} \dot q_1 = S' - r(q_1 - q_2) - \mu_1 q_1 \\ \dot q_2 = r(q_1 - q_2) - \mu_2 q_2 \end{cases} \iff \begin{cases} \dot\alpha_1 = \\ \dot\alpha_2 = \end{cases}$$

(q_1/V_1 - q_2/V_2)

Mixing problems:

#2: Derive eq-ns (DS) for concentrations $\{\alpha_1(t), \alpha_2(t)\}$ in Exchange system 3.

#3 (special case of exchange: $\mu_1 = \mu_2 = \mu$)

(i) Show that total amount $q = q_1 + q_2$ obeys (DE) $\boxed{\dot{q} = S - \mu q}$. Solve it for $q(t)$.

(ii) Use solution $q(t)$ of (i) to get a single DE for $\boxed{\dot{q}_1 = \dots}$ (Hint: $q_2 = q - q_1$)

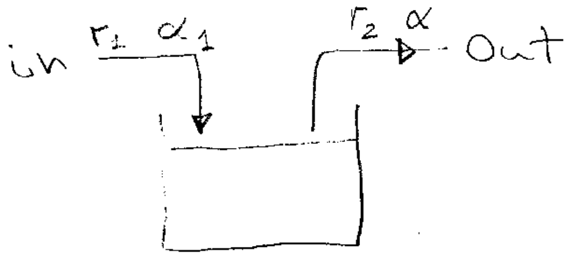
Write DE (solution will come later)

#4 (Physiology) Medication (q) is supplied intravenously (IV) at flux rate $r_1 = .02$ L/min concentration $\alpha_1 = 5$ mg/L for 20 min. It degrades in blood at rate $\mu_1 = .3$/min, and is removed completely by kidney. Given blood volume $= 5.5$ L and exchange rate (blood ⟷ kidney) $r = .2$ L/min. Set up DE model for $q(t)$ (w. diagram) and solve it. Find drug concentration 2 hr after injection.

Excess fluid (injected by IV) is also removed by kidney

Mixing system

1



r = rate $\left[\frac{cm^3}{sec}\right]$

α = concent $[gr/cm^3]$

$q = [gr]$

V = volume $[cm^3]$

$q = V\alpha$ - quantity

Balance:

1) $\Delta V = (r_1 - r_2)\Delta t \Rightarrow \boxed{V' = (r_1 - r_2)}$

2) $\Delta q = \left(\alpha_1 r_1 - \frac{q}{V} r_2\right)\Delta t$

$$\Rightarrow \boxed{q' + \frac{r_2}{V} q = \alpha_1 r_1}$$

1° Equal rates $\boxed{r_1 = r_2} \Rightarrow V = V_0$ - const

$$\Rightarrow q(t) = \underbrace{\frac{\alpha_1 r_1}{r_2/V}}_{\alpha_1 V} + \Big(q_0 - \underbrace{\frac{\alpha_1 r_1}{r_2/V}}_{\alpha_1 V}\Big)e^{-\frac{r_2}{V}t}$$

2° Zero incoming conc. $\boxed{q' + \frac{r_2}{V} q = 0}$

$\alpha_1 = 0$

$$\Rightarrow \boxed{q(t) = q_0\, e^{-\int \frac{r_2}{V} dt}}$$

3° Different rates:

$V = V_0 + (r_1 - r_2)t$

$\boxed{q' + \frac{r_2}{V} q = \alpha_1 r_1} \leftarrow$ Non separable (Linear DE)