STOCHASTIC MODELS IN ENGINEERING AND SCIENCE
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Geostrophic turbulence: How Jupiter got its stripes?

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Applied Math projecyts at Case in 90s

- Center for Stochastic and Chaotic Processes
- ONR project on "Stochastic modeling in oceanography"
- CRDF projects on "Geophysical modeling"
  - V. Klyatskin
  - **S. Danilov**
  - F. Dolzhansky

} IAP, Moscow, Russia

Topics:

- Methods of Statistical topography in Turbulent transport

1. **V. Klyatskin, D. Gurarie, W. Woyczynski**, *Diffusing passive tracers in random incompressible flows: statistical topographic aspects*, **J. Stat. Phys.**,1996, 84, #3/4, 797-836.
2. **D. Gurarie, V. Klyatskin, W. Woyczynski**, *Short-time correlation approximations for diffusing tracers in random velocity fields: a functional approach*, Stochastic modeling in physical oceanography, **Progress. in Prob.**, 39, Birkhauser, Boston, Boston, MA, 1996, 221-269, 1997
3. V. Klyatskin, D. Gurarie, *Coherent phenomena in stochastic dynamical systems*, Physics - Uspekhi Fizicheskikh Nauk (**Russian Physics Surveys**), 169, no.2, 1999, 171-207
4. ….

Lecture Plan

Geophysical flows

rotation and gravity

Vorticity conservation and Quasigeostrophy

Turbulence

3D and 2D phenomenology (direct and inverse energy cascades)

Rhines theory

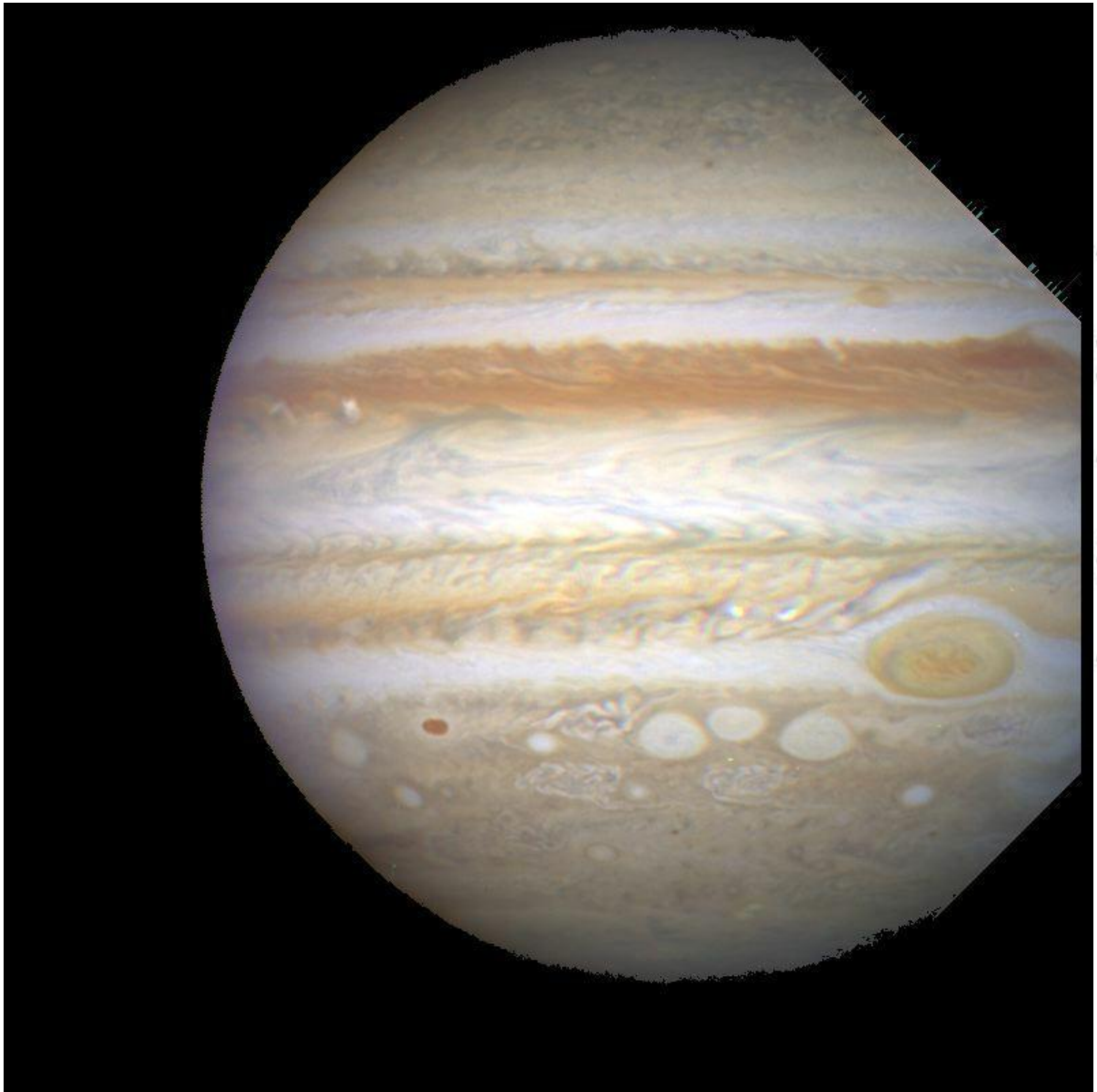
CFD “turbulence”

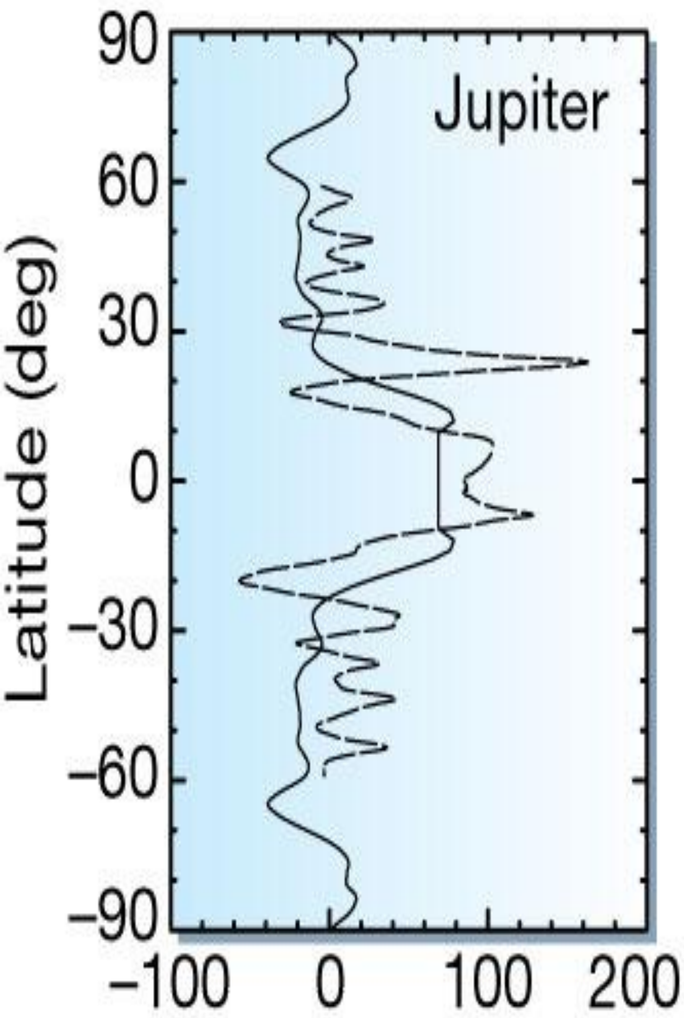
Spectra and large-scale structure (jets and vortices)

Evolution of zonal jets

Open problems and issues

Zonal jets on Jupiter



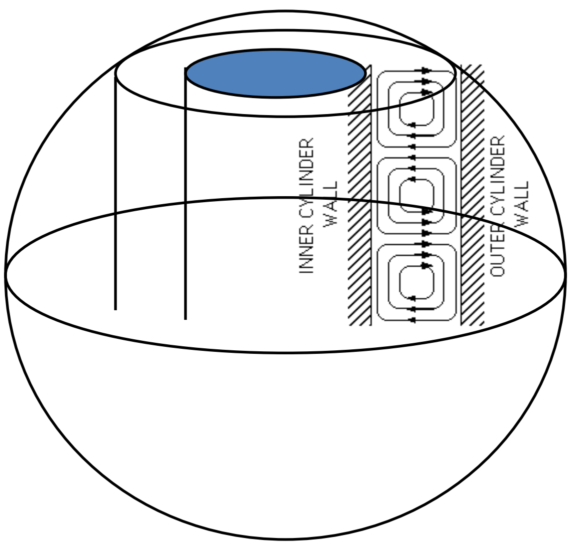


Zonal winds m/s
Dashed - measured
Solid - simulated

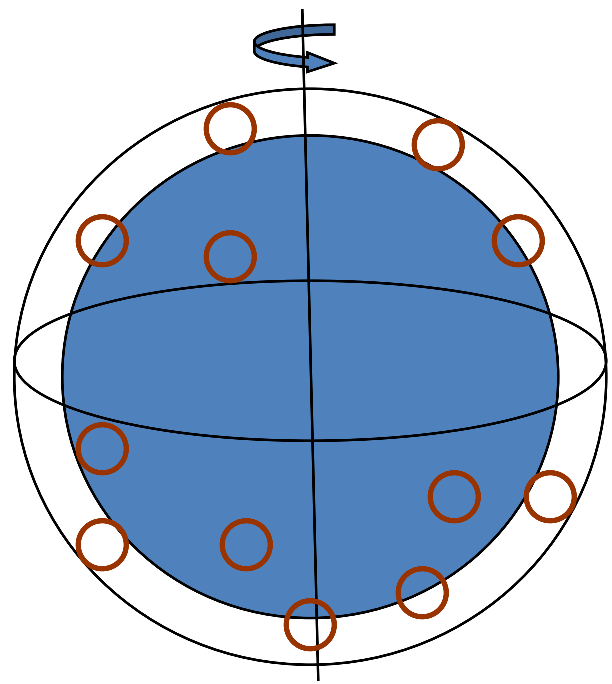
Origin, mechanism ??

Taylor-Couette vortices in differentially rotating spherical (cylindrical) shells (Busse, 1983)

Quasigeostrophic 2D-turbulence, stirred by “small-scale” stochastic forcing (Rhines 75, Wilson 78, …)



Deep convection



Shallow convection

I. Basic fluid equations

Velocity : $\vec{v} = (u, v)\ (2D)\ (u, v, w)\ (3D)$

Stream : $\psi(x, y) \rightarrow \vec{v} = (-\psi_y, \psi_x)\ (2D)$

Vorticty : $\zeta = v_x - u_y = \nabla^2 \psi$

Momentum conservation : $\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{v} + \vec{F}$

Incompressibility / mass : $\nabla . \vec{v} = 0$

Energy cons. ...

p - pressure; ν - viscosity; F - body forces

Vorticity form :

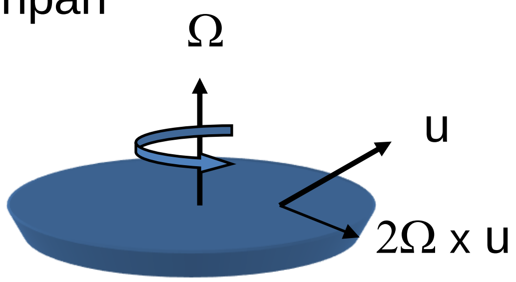
$$\partial_t \zeta + J(\psi, \zeta) = 0 \text{ or } \nu \nabla^2 \zeta,$$

$$J(f, g) = det \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$ - Jacobian

Solutions ? Analysis ?

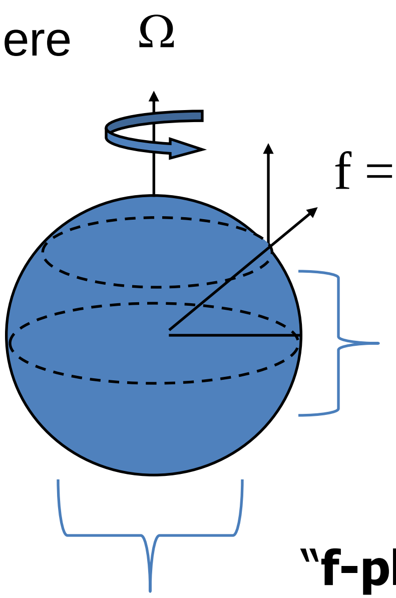
Rotating fluids: Geostrophic balance

Dishpan



$f=2\Omega$ – local Coriolis

Sphere



$f = 2\Omega\cos(\theta)$

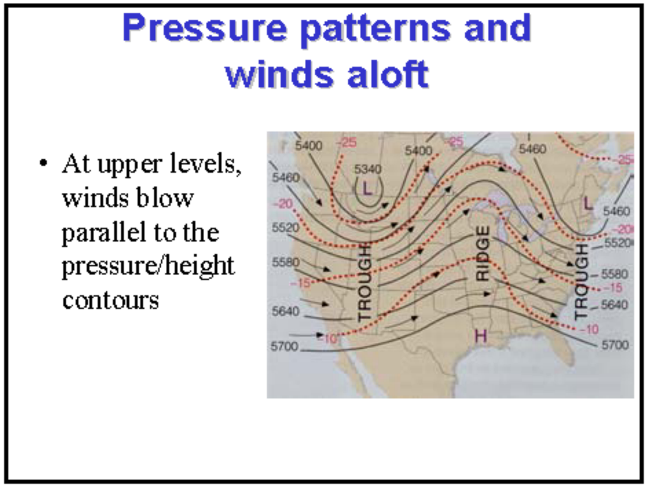
$$\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \vec{f}\times\vec{u} = -\frac{\nabla p}{\rho} + \nu\nabla^2\vec{u}$$

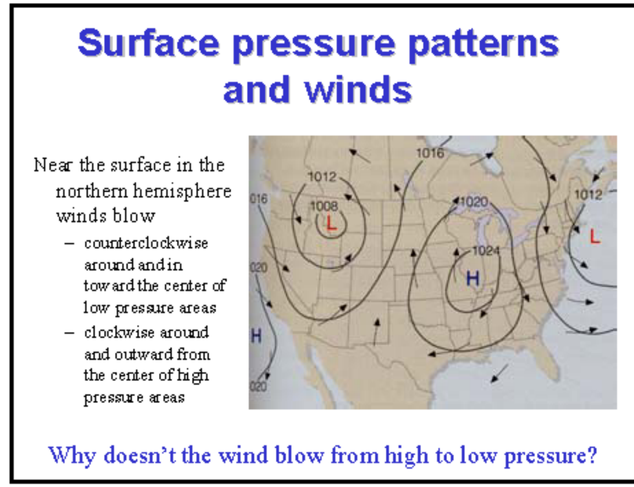
Geostrophic winds follow isobars

"β-plane" $f \approx \beta y$

"f-plane" $f \approx f_0$

Geostrophic winds follow isobars





Dominant factors/forces for large scale geophysical flows

- Rotation (Coriolis, geostrophic balance)
- Gravity (stratification, hydrostatic pressure)

Word ***geostrophic***

made of two Greek roots:

Geo *(earth)* + ***Strophe*** (part of an ancient Greek choral ode sung by the chorus when moving from right to left, hence movement performed by the chorus during the singing of this part) – *cyclonic motion*

Ancient Greeks were interested in meteorology, and several examples of ***meteorological terms*** with Greek roots survived, though not their ***theories*** (Aristotle thought that winds from the west are cold because "*they blow from the sunset*")

Quasi-geostrophy (QGS)

- QGS = departure from Geostrophic balance
- **Method**: rescaling/expansion in small *Rossby number Ro=U/(f L)* (similar to "Mach" in compressible gas) for linearized PV about *geostropic balance*, dropping "fast" (gravity) modes
- Rossby on Jupiter: $.02<Ro<.3$

Conservation of Potential vorticity: derivation of quasigeostrophy

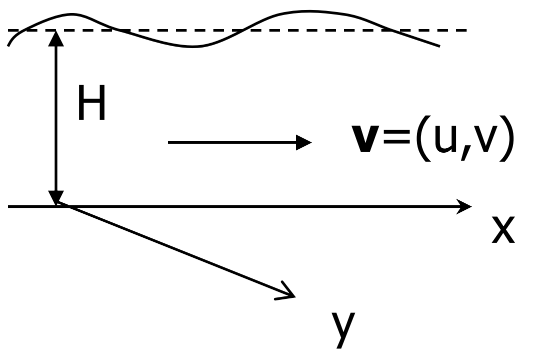
Primitive (mass-momentum) equations conserve ***Potential Vorticity***

PV = "planetary f" + "relative" $\zeta = \nabla \times \mathbf{v}$

Example:
Rotating Shallow Water on f-plane

$$D_t \mathbf{v} + \mathbf{f} \times \mathbf{v} = -g\nabla h + \ldots$$

$$h_t + \nabla \cdot \left[\left(H + h \right) \mathbf{v} \right] = 0$$



g – gravity acceleration = 9.8m/s²
H – mean height/depth
h – departure from mean

$$\text{PV: } q = \frac{f + \zeta}{g\left(H + h\right)} \approx \frac{f}{gH}\left(1 + \zeta / f - h / H\right)$$

- ***Quasigeostrophy*** – small departure from ***geostrophic balance***
- Expansion in small Rossby number Ro (.02<Ro<.3 for Jupiter!) for ***linearized* PV**

Examples of QGS

$q = \frac{f+\zeta}{g\,h}$ (shallow water PV or its linearization)

$$\partial_t q + J(\psi, q) = D q + F$$

2 D (Euler) fluid :	$\zeta = \nabla^2 \psi$	$\partial_t \zeta + J(\psi, \zeta) = D[\zeta] + F$
Rotating dish pan "f-plane" :	$q = \left(\nabla^2 - 1/L^2\right)\psi$	$\partial_t q + J(\psi, q) = D[q] + F$
Rotating sphere of radius R :	$Q = \nabla^2 \psi + R\,\mathrm{Cos}(y/R)$	$\partial_t Q + J(\psi, Q) = D[Q] + F$
Equatorial channel "beta-plane" :	$q = \nabla^2 \psi + \beta y$	$\partial_t \nabla^2 \psi + J\left(\psi, \nabla^2 \psi\right) + \beta \partial_x \psi = D \psi + F$

F – Forcing; $D = \lambda + \nu \nabla^2$ - "friction

II. Turbulence

Onset: shear instability

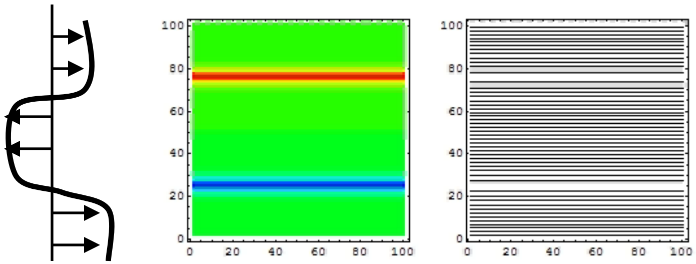
Shear profile : $U(y) = -\Psi_y$; $Z(y) = -U_y$;

Perturbation : $\psi = \Psi(y) + \psi'$; $\zeta = Z(y) + \zeta'$;

Linarized problem : $\partial_t \zeta' + L[\zeta'] = 0$

Linear Operator : $L = [(\beta + U_{yy})\nabla^{-2} - U)]\,\partial_x - D$

Kelvin-Helmholtz rolls



Planetary vortex (beta term) creates effective “shear”: $Z(y) = \beta\, y$

Kolmogorov-Obukhov theory (1941)

- ***Scales and Mode*** : fluid flow viewed as system of interacting (Fourier) modes
- ***Energy cascade***: energy distributed among modes (spectrum E_k), and exchanged (energy flux ε)

Premises:

•Viscosity is important only at small scales, hence large/ intermediate ones (***inertial range***) are “inviscid” (no energy loss)

•Mode interaction cascades energy from larger to smaller scales - ***direct cascade***

• In the absence of “distorting forces” energy spectra E_k should depend on scalar wave-number k (***isotropic spectrum***)

• ***Locality***: only nearby scales (modes) exchange energy, hence constant (scale independent) ***energy flux*** ε

• ***Scaling laws*** for E_k ?

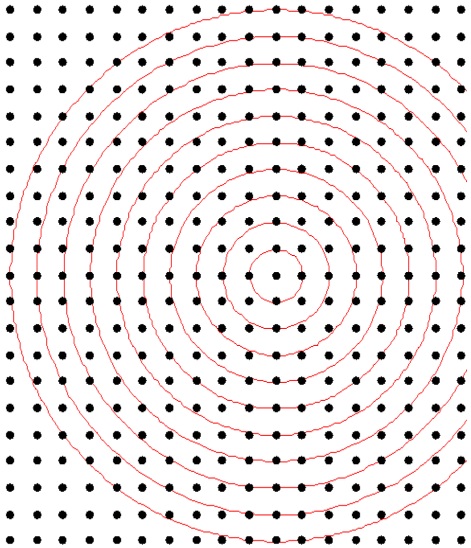
Mode decomposition

PDEs (Euler, Navier-Stokes et al) converted to system of interacting modes

$$\psi(\mathbf{x}) = \sum_{\mathbf{k}=(k,l)} \psi_{\mathbf{k}}\, e^{i\,\mathbf{x}.\mathbf{k}}$$ – Fourier modes (scale = | k |)

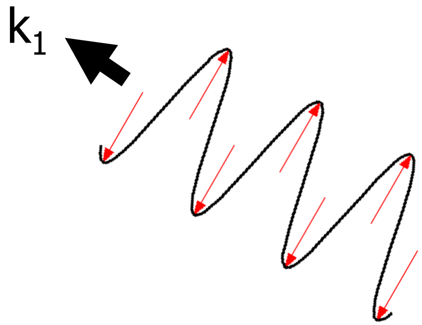
2 D Euler : $$\dot{\psi}_{\mathbf{k}} + D_{\mathbf{k}}\,\psi_{\mathbf{k}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} A_{\mathbf{p}\,\mathbf{q}}\,\psi_{\mathbf{p}}\,\psi_{\mathbf{q}} + F_{\mathbf{k}}$$

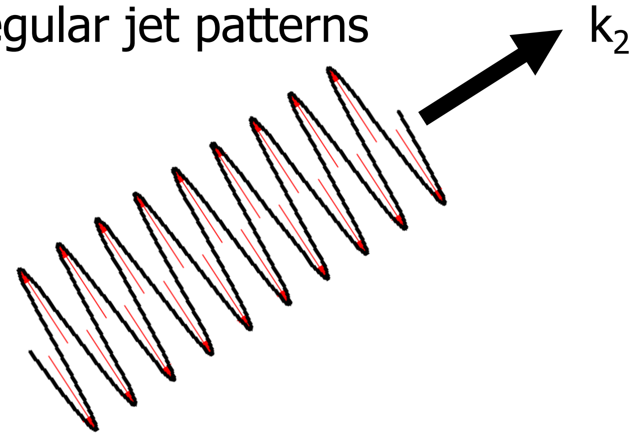
Coefficients : $A_{\mathbf{p}\,\mathbf{q}} = \dfrac{\mathbf{p} \wedge \mathbf{q}\,(p^2 - q^2)}{k^2}$ – interaction, $D_{\mathbf{k}} = \lambda + \nu k^2$ – dissipation

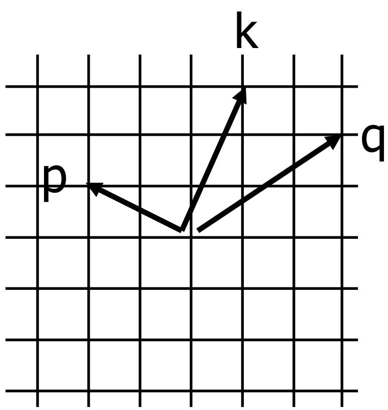


Spectral energy shells (2D)

Fourier modes = regular jet patterns







PDE converted to coupled ODES for mode coefficients:

$$a_{km}(t)\cos(kx+my)+\ldots$$

Dimensional argument [K41]

$$\text{Energy fux: } [\varepsilon] = \frac{m^2}{s^3};\ \text{Energy density:} \left[E_k\right] = \frac{m^3}{s^2};\ \text{Wave number:} \left[k\right] = \frac{1}{m}$$

$$\text{Scaling law: } E_k \sim C\varepsilon^p k^q \Rightarrow p = \frac{2}{3}; q = -\frac{5}{3}$$

$$\boxed{E_k = C_K \varepsilon^{2/3} k^{-5/3}}$$

Development

•Theoretical approaches:

moment closure scheme (quasi-normal, etc.), RNG, stochastic models

• Fluid experiments and observation

• Numeric simulations

•Applications of turbulence/scaling laws:

engineering, atmospheric turbulence, transport, diffusion etc.

2D turbulence phenomenology:
BKL (Batchelor-Kraichnan-Leith)

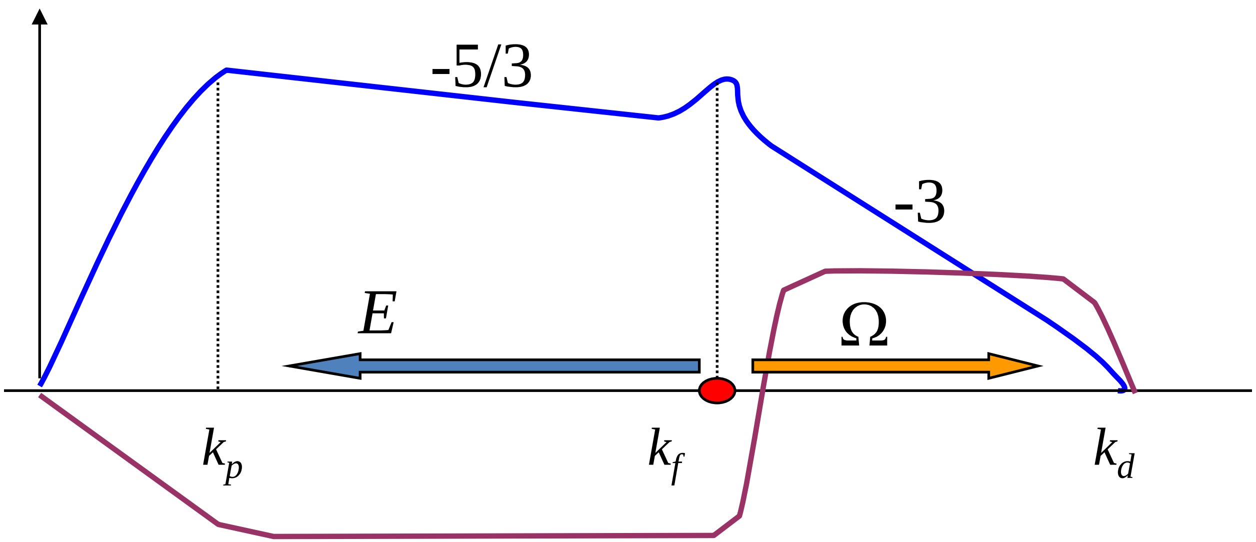
2D fluid (Euler) has many conserved integrals: "energy" + "moments of vorticity"

$$E = \frac{1}{2}\iint \mathbf{v}^2 dA = \sum_{\mathbf{k}} \left|\hat{\mathbf{v}}_{\mathbf{k}}\right|^2 \text{ - energy}$$

$$Z = \frac{1}{2}\iint \zeta^2 dA = \sum_{\mathbf{k}} k^2 \left|\hat{\mathbf{v}}_{\mathbf{k}}\right|^2 \text{ - enstrophy}$$

=> Two (possible) inertial ranges and cascades:
Direct cascade: "large" to "small" scales (enstrophy conservation)
Inverse cascade: "small" to "large scales" (energy conservation)

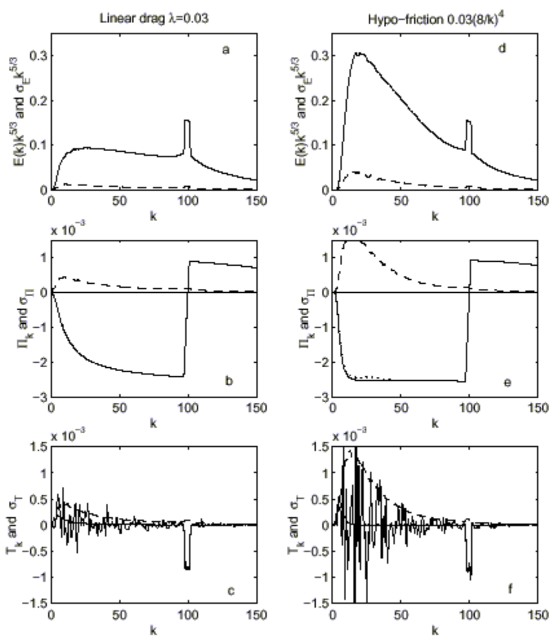
Isotropic Energy spectrum in 2D with source at k_f



Direct cascade:	Inverse Cascade:
"Big whorls have little whorls, Which feed on their velocity, And little whorls have lesser whorls, And so on to viscosity." (L. Richardson)	*"Big whorls meet bigger ones And so it tends to go on By merging they grow bigger yet And bigger yet, and so on…"* (M.E. McIntyre)

Inverse cascade leads to "large scale structures" in 2D flows !

Computational turbulence:
Departure from BKL phenomenology



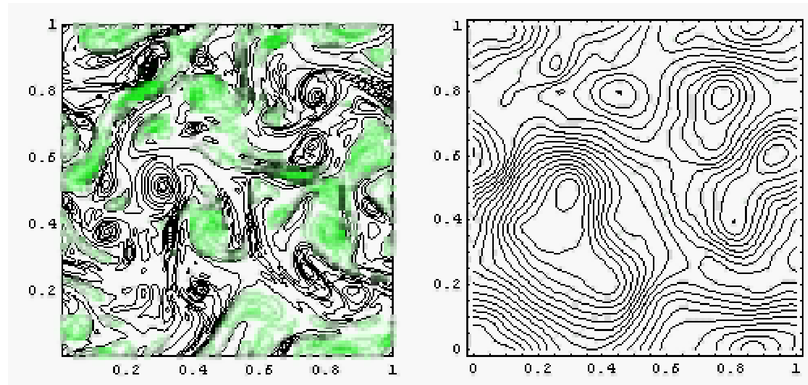
Numeric simulations on 512x512 spectral grid

- Steep spectra (slope > 5/3)
- Nonuniform fluxes
- Nonlocal transfers

•Vallis-Maltrud, 1991,1993
•Borue, 1994
•Sukhoryanski et al 1999
•Boffetta et al, 2000
•Danilov-Gurarie, 2001-2002

Computational turbulence: vorticity and stream field

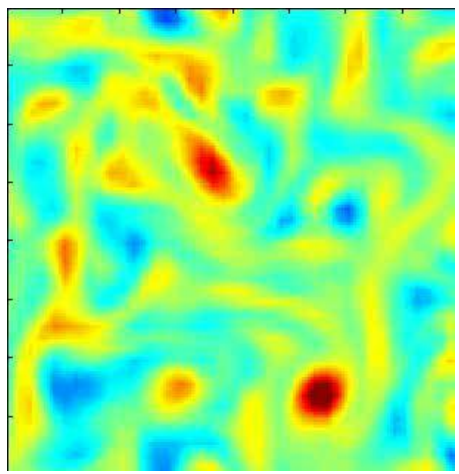
I. Semi-Lagrangian code (DG) on 150x150 grid
(spatial discretization, and vorticity conservation)

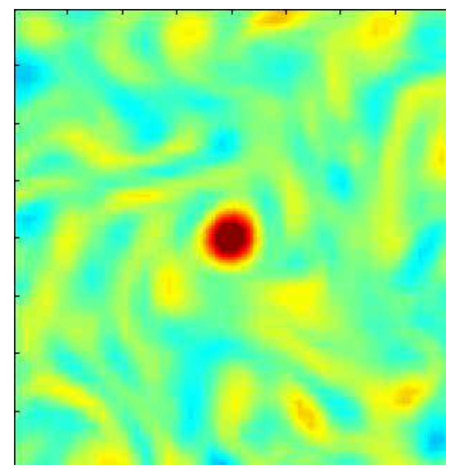


Stochastically forced turbulence (at k_f=80)

Strong vortex in forced turbulence

II. Spectral code (SD) on 512x512 Fourier grid
Fast FFT





- ***Inverse cascade*** leads to **vortex mergers**
- Can it lead to **strong jets ?**

III. Rhines theory of "beta-plane" turbulence

Beta-plane QGS combines "linear dispersion operator" + "nonlinear HD"

$$\partial_t \nabla^2 \psi + \underbrace{J(\psi, \nabla^2 \psi)}_{\text{NL term}} + \underbrace{\beta \partial_x \psi}_{\text{Linear}} = D\psi + F$$

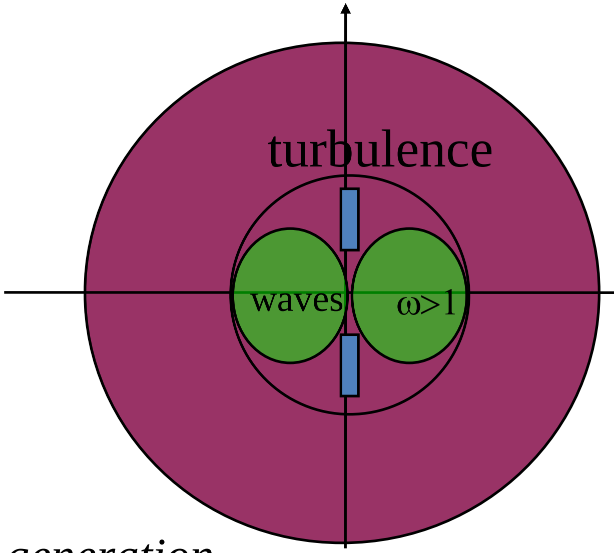
Imposes additional constraint on triad interactions ("weak turbulence")

$$\partial_t \psi_{\vec{k}} + i\,\omega_{\vec{k}} \psi_{\vec{k}} + \sum_{\vec{p}+\vec{q}=\vec{k}} A_{\vec{p}\,\vec{q}} \psi_{\vec{p}} \psi_{\vec{q}} = D_{\vec{k}} + F_{\vec{k}};$$

$$\omega_{\vec{k}} = \frac{\beta k_1}{k_1^2 + k_2^2} \quad \text{– Rossby dispersion (waves)}$$

$$\vec{k} = \vec{p} + \vec{q} \quad \text{– triad interaction}$$

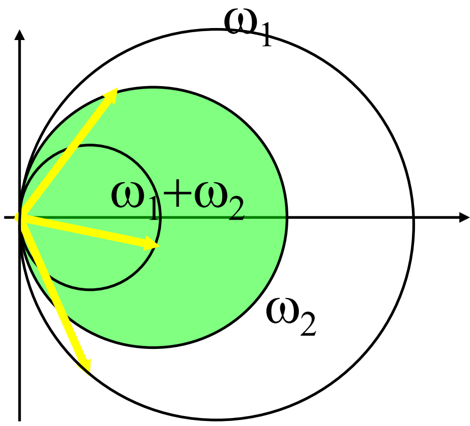
$$\omega_{\vec{k}} = \omega_{\vec{p}} + \omega_{\vec{q}} \quad \text{– wave resonances}$$



- "Beta" inhibits inverse cascade at scale through *Rossby wave generation*
- Suppresses cascade into high Rossby region ("lazy eight") via "frequency detuning"
- Channels energy into zonal (near zonal) modes

Frequency-detuning mechanism

(Holloway-Hendershot, 1977)



- Cascade into lazy-eight imposes frequency synchronism
- Triad transfers are dampened by Rossby-wave dispersion

Rhines phenomenology:

"Eddy time-oscillations" = "***Rossby frequency***"

coupled with "Eddy speed" = "***Rossby phase speed***"
yield "***Rhines scale***" = "arrest scale" for inverse cascade on beta-plane

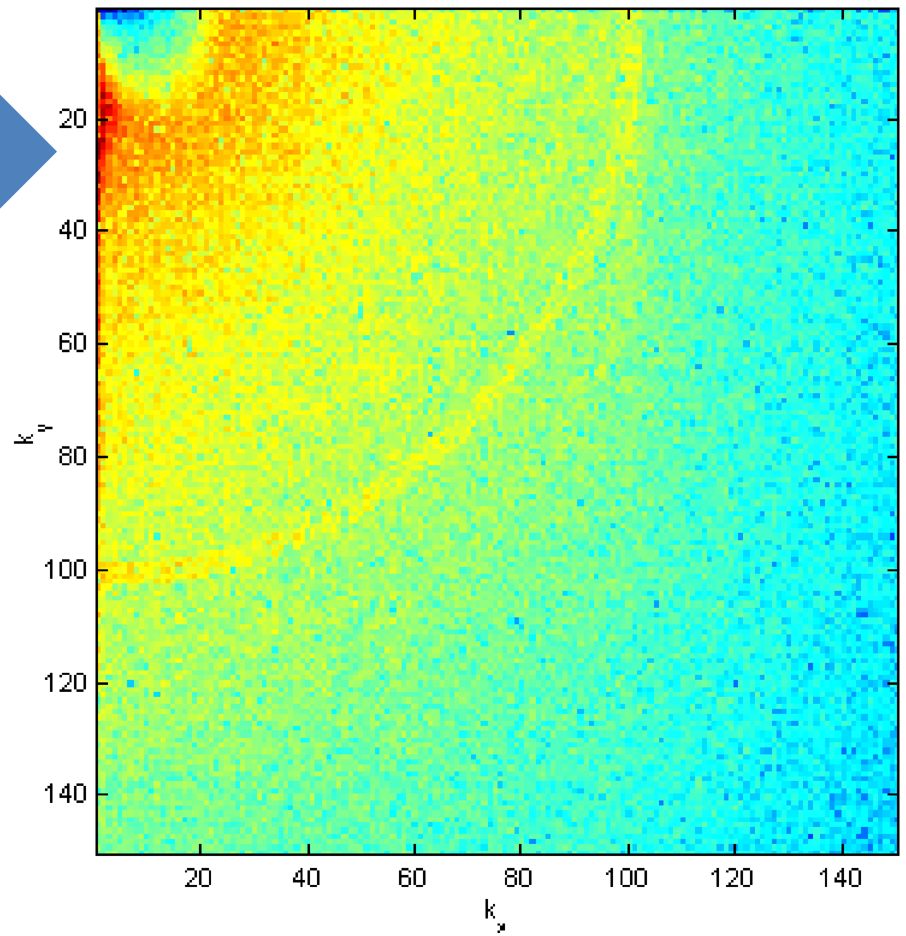
$$k_{Rh} = \sqrt{\frac{\beta}{2\, U_{rms}}} = \frac{\beta^{1/2}}{2\, E^{1/4}}$$

Also Rhines predicted k^{-5} zonal spectrum at large scales (?)

Computational work: DNS on beta-plane (…; DG)

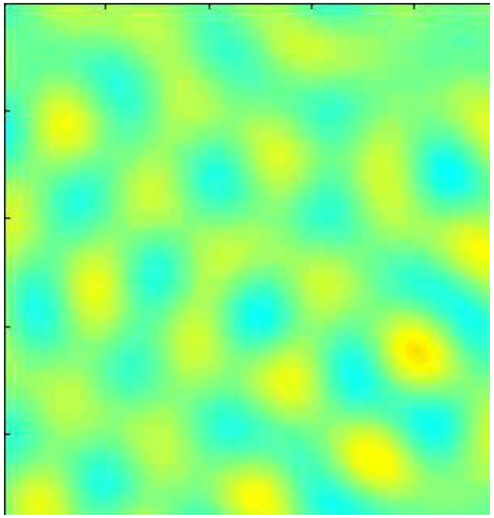
Beta-plane spectrum

Spectral anisotropy - accumulation of zonal modes

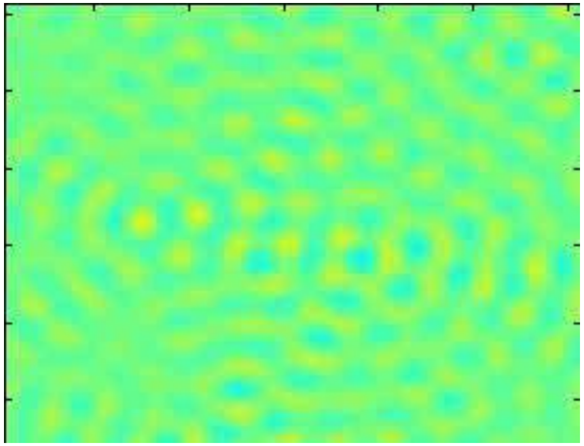


Zonal jets: initial growth phase

Isotropic case

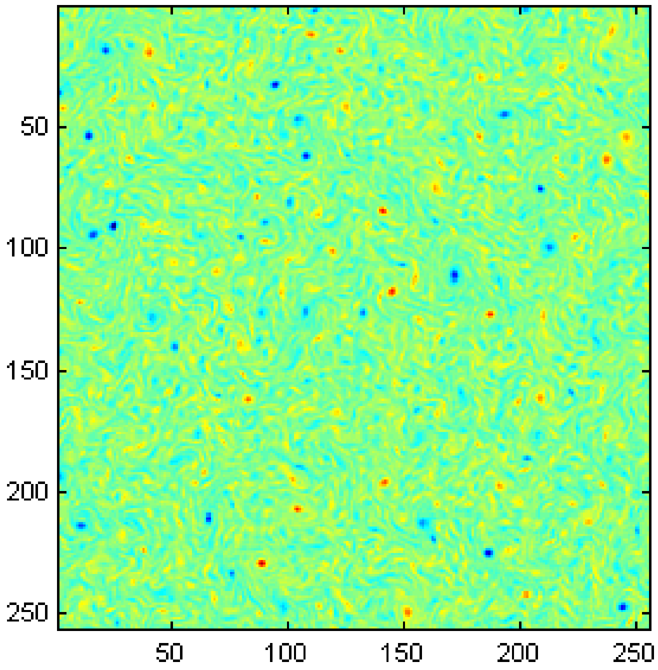


Beta- plane



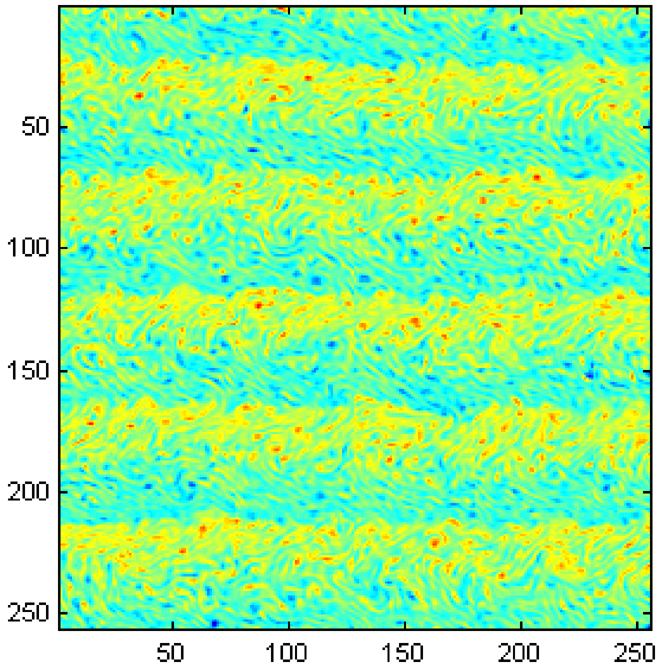
Typical vorticity fields (DG)

Isotropic (β=0)



Strong localized vortices

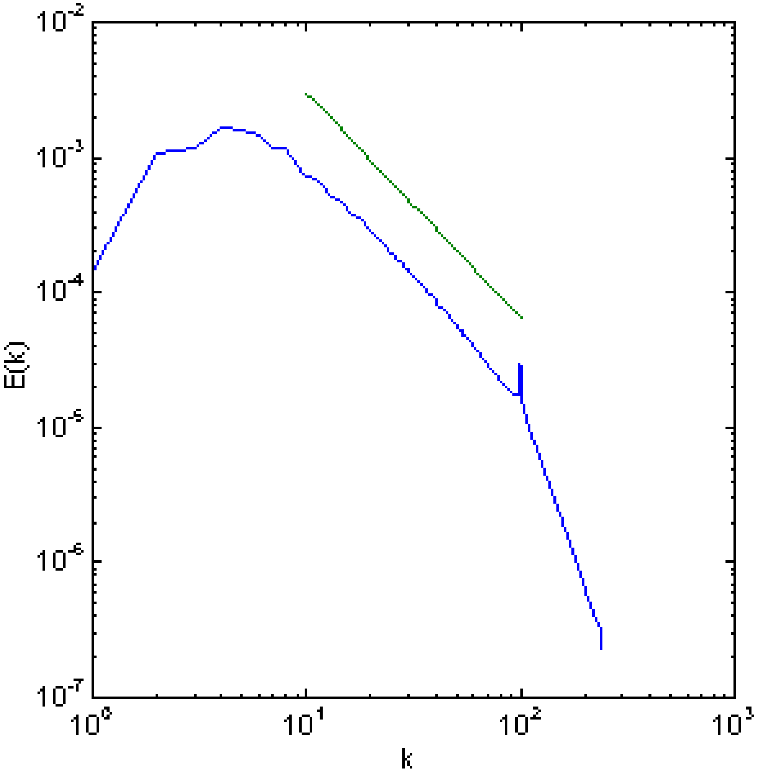
Beta



Zonal jets

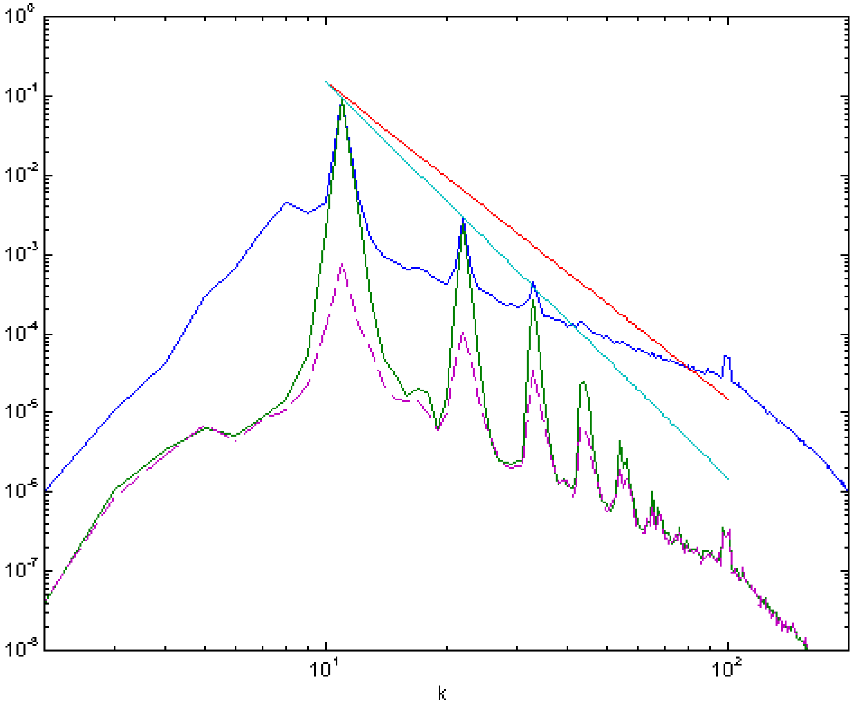
Typical energy spectra (DG)

Isotropic



5/3 slope

Beta

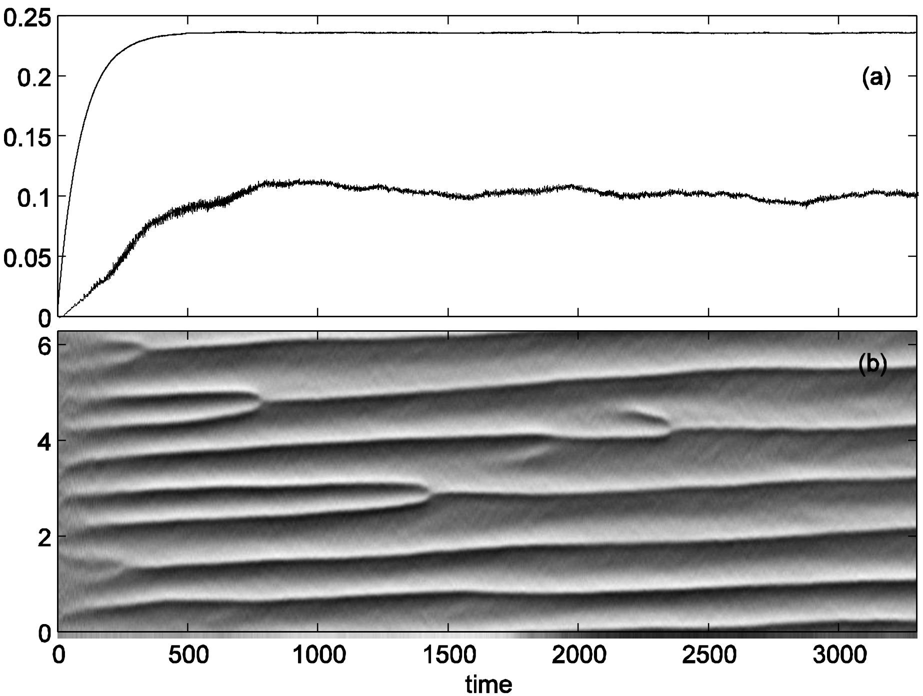


-4 and -5 slopes

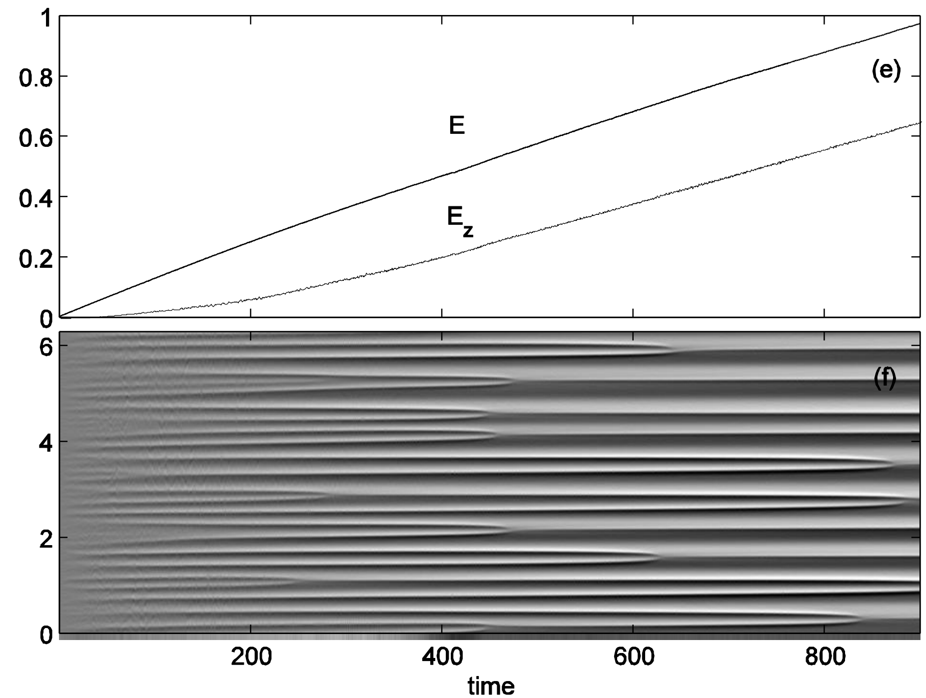
Dominated by zonal spectral peaks

Long term evolution of zonal jets

Typical friction case



Frictionless case



Slow emergence and evolution through jet mergers

Stabilizes at k_{Rh}

Vorticity field and Zonal profiles (Jovian jets ?)

Frictional dissipation

“slope”/beta

Hypofriction
β =320

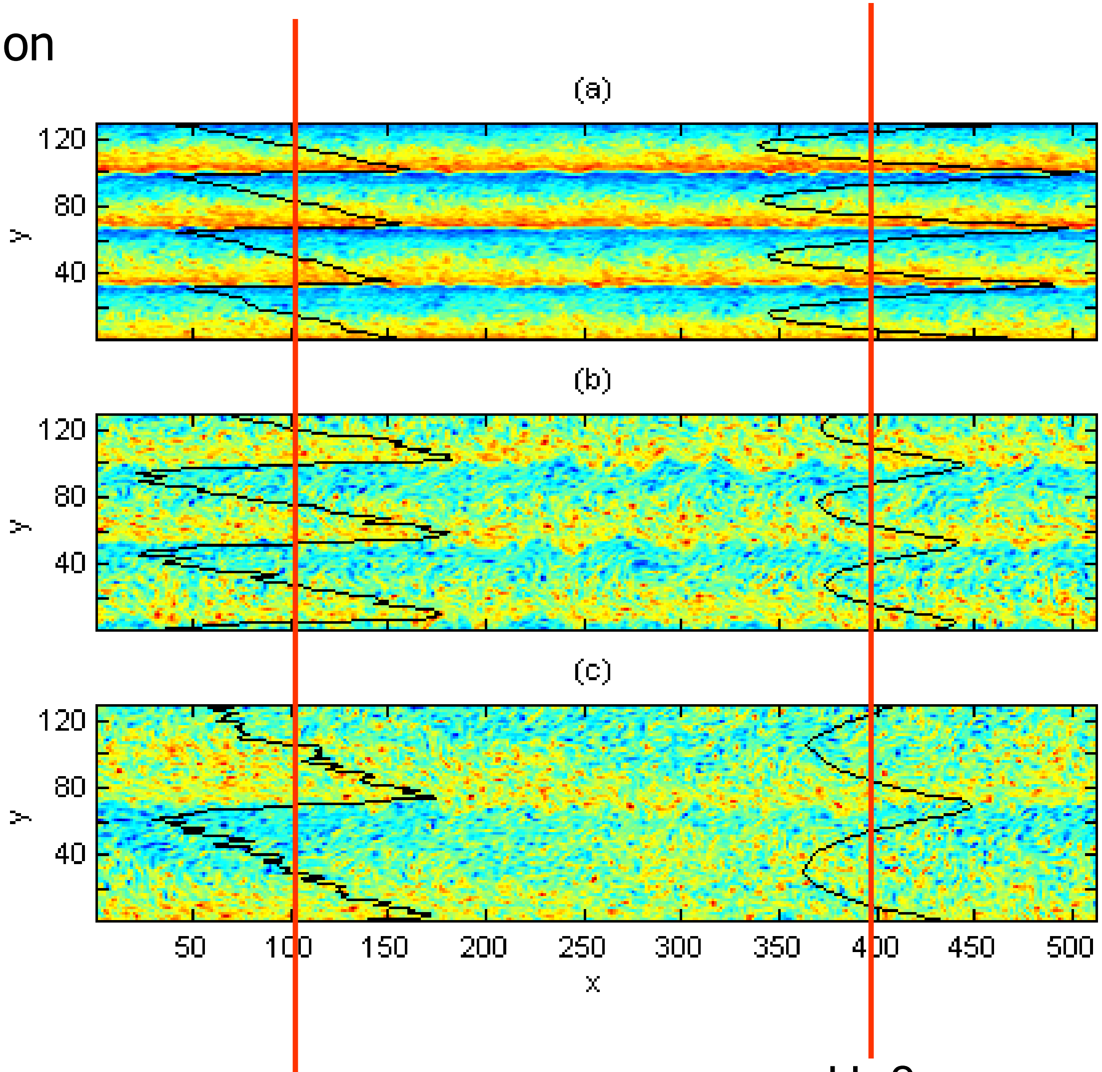
= .6

Hypofriction
β =80

.5

Standard
(bottom) friction
β = 80

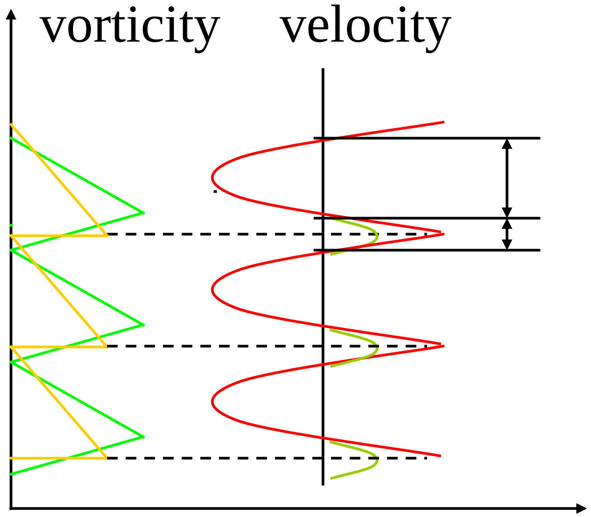
.2

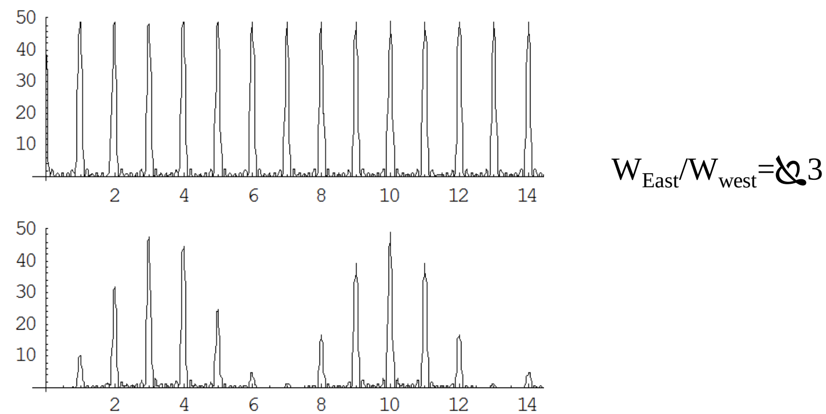


Vorticity $\zeta=0$

U=0

Idealized zonal profiles and “scaling laws”





Model shape functions:

$$\Phi_N(x) = \{ \; |\sum_{m=0}^{N} e^{2\pi i m x}|^2; \quad \text{saw}$$
$$| \sum^{N} (e^{2\pi i m (x-a)} - e^{2\pi i m x}) |^2; \quad \text{slopi}$$

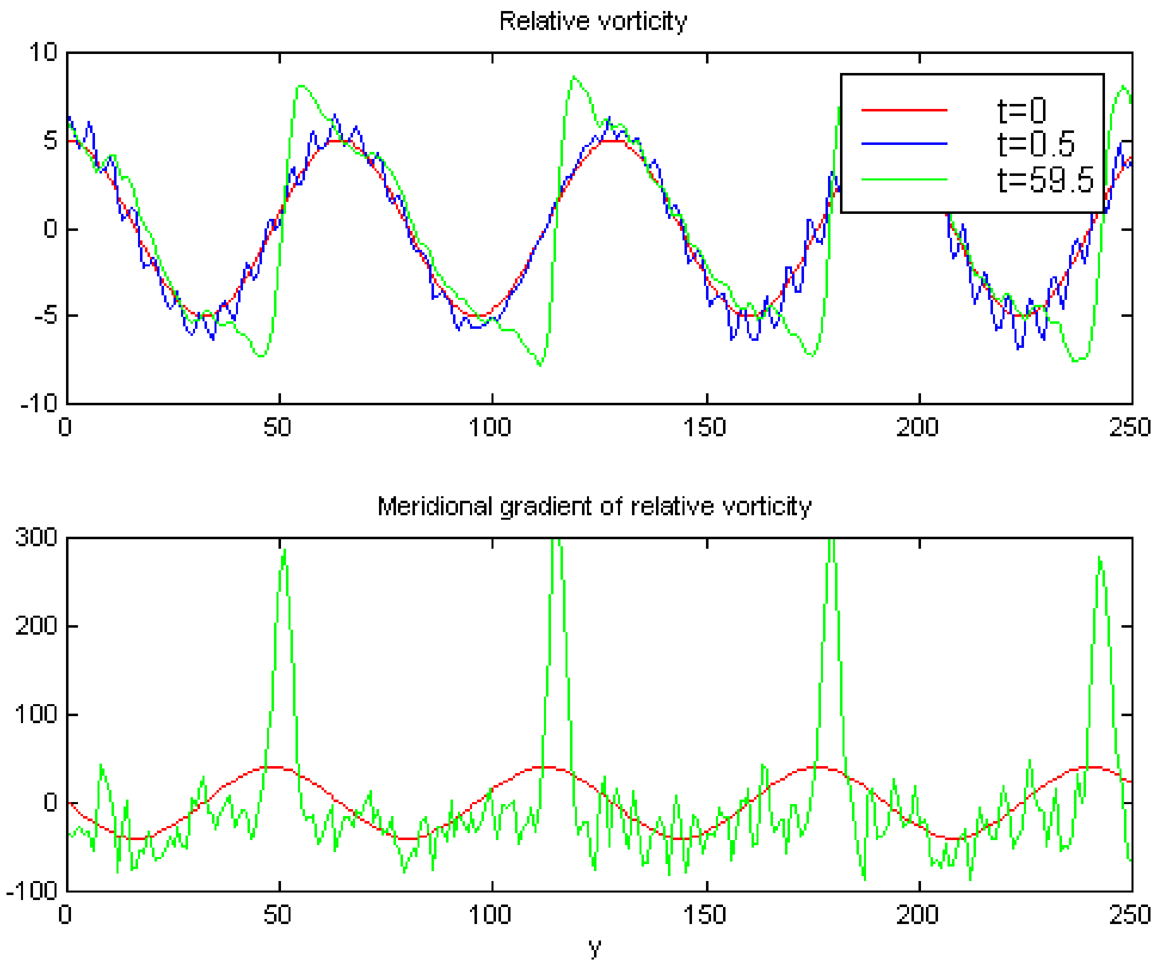
Zonal spectra (based on “shape function”)

$$E_{Zon}(k) = \frac{C\,\alpha^2\,\Phi(k/k_{Rh})}{k^p\,k_{Rh}^{\,q}};$$

- “***Scaling laws***” for zonal spectra (peaks) have sensitive dependence on “jet patterns”. Could vary between $k^{-4} – k^{-6}$.
- Proposed “***k^{-5} law***” is special case (nothing “universal”)

"Stare-case" zonal attractor

Initial symmetric profile $\zeta = \sin(k\ y)$ with $k = k_{Rh}$ develops "stare-case" pattern



Theory of slow zonal evolution / equilibration:

Manfroy-Young (1999), Stuhne (2001)

Premises/method:

•"Beta effect" on transverse (meridional) jet in weakly "supercritical regime" (past "linearized stability")

•Expansion in "small parameter" = supercriticality (above bifurcation level)

•"Zonal jet equation" arises as slow (modulation) amplitude DE of Cahn-Hillard type.

Interesting ideas, but …

• Premises do not apply to strong turbulent regime with forcing/dissipation.

• Zonal jets don't grow as "*instabilities*" of large scale regular (meridional) flow, but arise as "*instabilities*" of small-scale turbulent "5/3 background"

• The proper theory/model - still wonting!

Conclusions

"… *the turbulence community has been facing crisis through emergence of* ***coherent structures*** *in numerical and laboratory modelings so that the standard theories based on* ***homogeneity*** *and* ***isotropy*** *are increasingly challenged…"*
(Quote from GTP/NCAR workshop on "Cumulus Parameterization in the Context of Turbulence", 2004)

• Stochastically forced 2D turbulence (Euler/ QGS) channels energy to large scales (Inverse Cascade).

•The large-scale coherent structures on beta-plane is array of zonal jets, at Rhines scale.

• Slow zonal evolution leads to a "stare-case" vorticity profile of (negative) slope: $-\alpha > -\beta$, and the corresponding "strong" eastward and "weak" westward jets.

• The "stare-case" pattern serves as an "attractor" of stochastically forced turbulent flows.

• Mathematical challenge: explain (develop model of) "zonal instability" of "5/3 turbulent background" that gives rise to jet structure, its slow evolution and persistence

Are Jupiter stripes “zonal jets” of QGS turbulence

(“Busse” vs. “QGS”)

???