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Coherent 'vortex solitons' in quasi-inviscid two-dimensional turbulence: analytic structure and stability

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Relaxation and equilibria of 2D fluids

Robust 2D structures with localized vorticity arise often in long-term simulations of the Navier – Stokes or Quasi-inviscid fluids, as they relax to a equilibrium state via *inverse cascade* mechanism (vortex mergers).

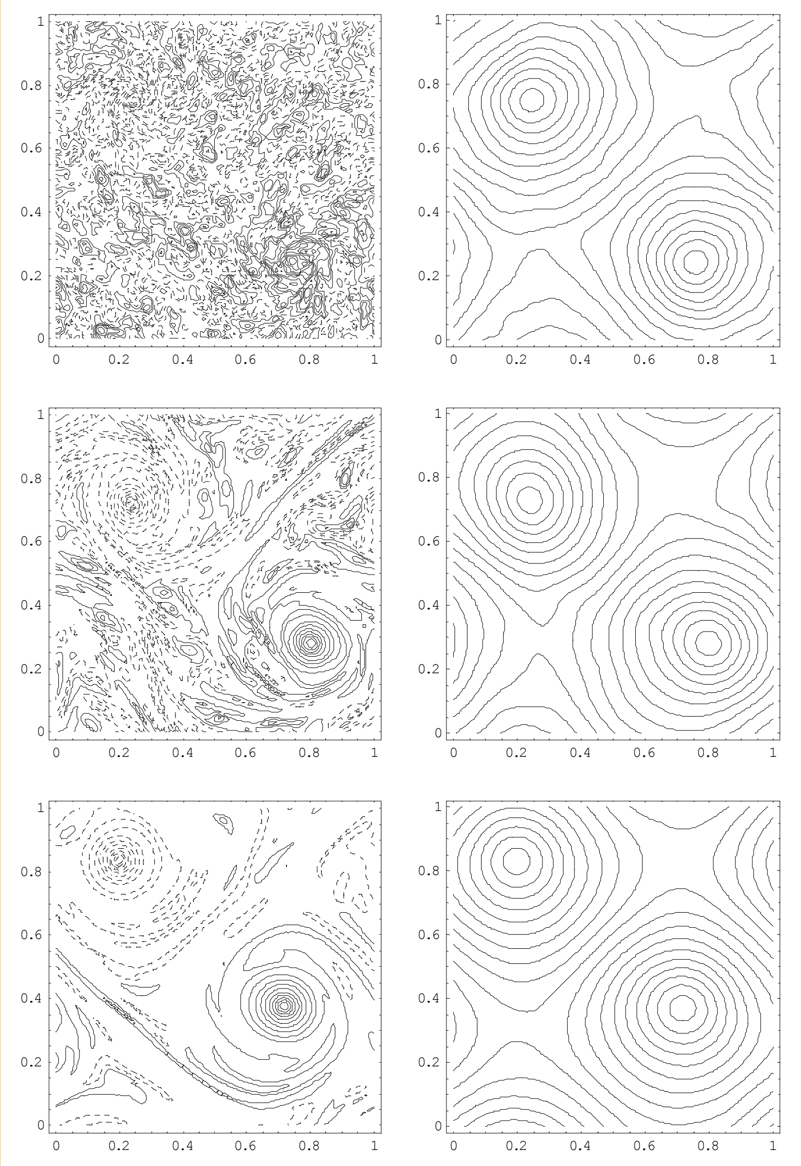


Fig. 1: *Typical inverse cascade (equilibration) of randomly perturbed vortex dipole in square geometry with double-periodic B.C. (vorticity-left, stream-field –right); initial, mid-term and final stage.*

Any stationary (equilibrium) flow of an ideal 2D Euler fluid obeys *stream - vorticity* relation with some function *f*:

$$\zeta = \nabla^2\psi = \psi_{xx} + \psi_{yy} = f(\psi)$$

Sinh-Poisson equation and vortex solitons

Among many choices, *f=Sinh* plays important role. It appears in *statistical theories* of 2D turbulence (both for vortex gas and continuous/patchy vorticity, e.g. [1-2]), as well as *numeric simulations* of Navier-Stokes or quasi-inviscid fluid (ref.[3-4]). Fig. below illustrates it for quasi-inviscid fluid simulated by a semi-Lagrangian code

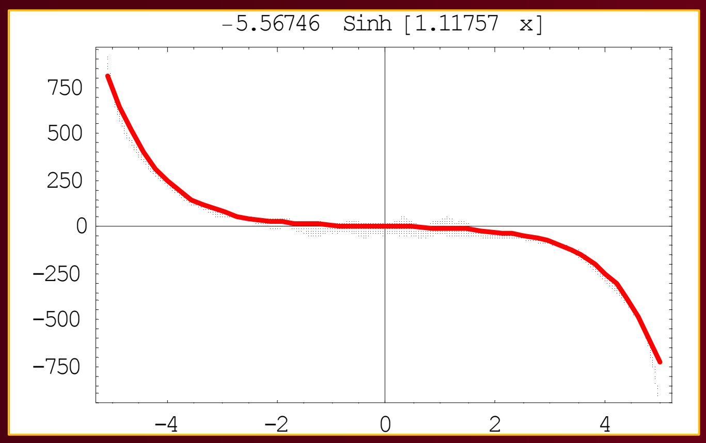


Fig. 2: *Typical sinh-Poisson (stream-vorticity) scatter plot produced in quasi-inviscid (semi-Lagrangian) simulations (D. Gurarie)*

Sinh-Poisson is closely related to integrable nonlinear models (sin-Gordon) and has large families of soliton-like solutions, both single-periodic (Mallier – Maslowe[5]), and double-periodic[6-7]. We call them 'vortex solitons'. All Sinh-solitons have the following analytic form:

$$\psi_S = 4\tanh^{-1}(\theta)$$

with 'phase' theta (either trigonometric or elliptic Jacobi). Examples:

$$\theta(x,y) = \left| \begin{array}{ll} \dfrac{\varepsilon\, Cos(y)}{Cosh(ax)} \text{ - Mallier-Maslowe} & \text{Single-periodic} \\[2ex] \dfrac{\varepsilon\, cn(ay\,|\,k)}{cn(ax\,|\sqrt{1-k^2})} & \text{Double periodic} \\[2ex] \dfrac{\sqrt{k}\ \mathrm{sn}(rx,k) - \sqrt{k_1}\ \mathrm{sn}(sy,k_1)}{1+\sqrt{kk_1}\,\mathrm{sn}(rx,k)\mathrm{sn}(sy,k_1)} & \text{arrays} \end{array} \right.$$

The last example, called sn-sn plays important role in relaxation of quasi - inviscid fluids. All known 'double-periodic soliton families', like sn-sn, depend on a single parameter, Jacobi modulus $0<k<k_0$, which measures their *degree of non-linearity* (k=0 corresponds to 'linear' Fourier modes): so 'small k' are 'near linear', large k - strong localized vortices

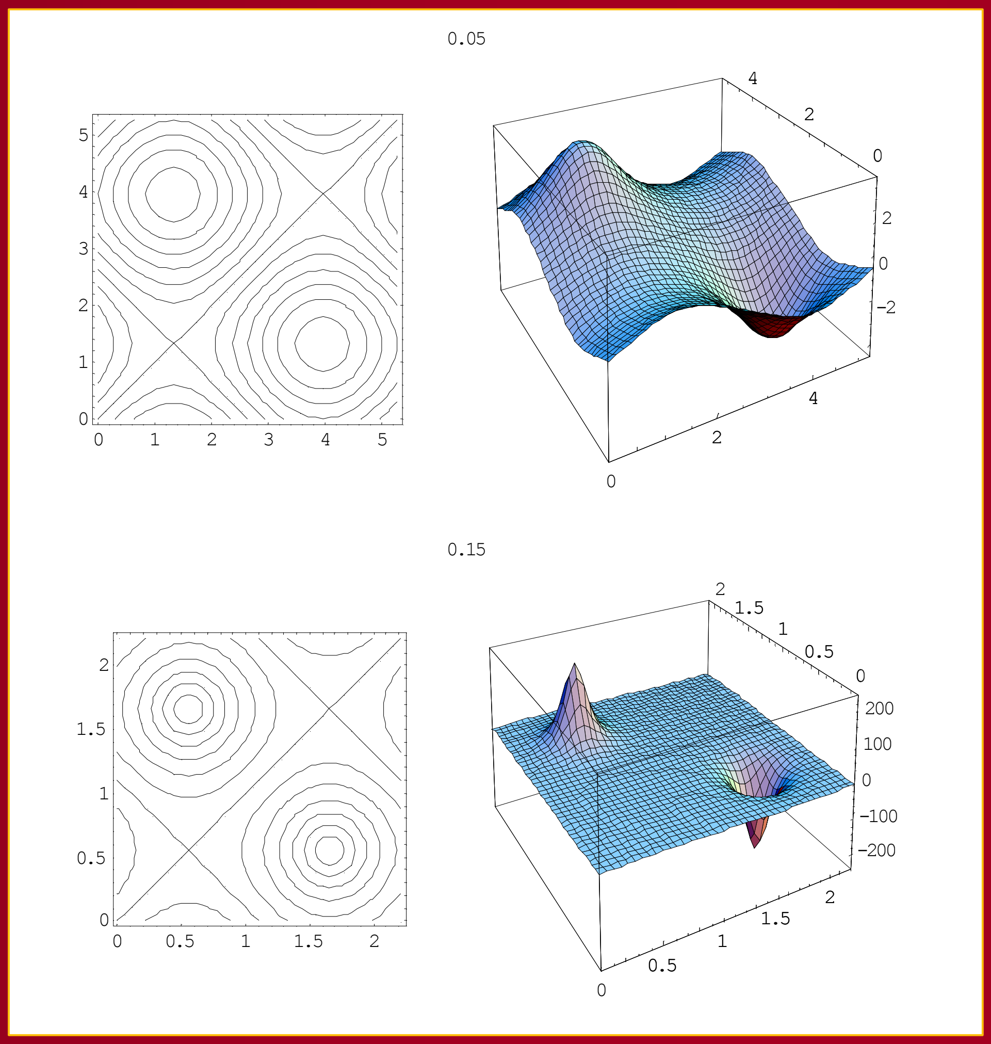


Fig. 3: *sn-sn vortex dipoles at two values k: 'small' k=.05 (near Fourier) 'large' k=.15 (strongly nonlinear)*

Basic problems:

(i) Are 'vortex solitons' realizable, as terminal states of decay turbulence (relaxation equilibria) of 2D fluid?

(ii) Are they stable: dynamically (for perturbations), by 'Arnold test' (as equilibria of ideal 2D Euler), other (??)

Quasi-inviscid (semi-Lagrangian) fluid

We study dynamic stability via quasi-inviscid semi-Lagrangian code (D. Gurarie). It advects vorticity along Lagrangian paths, but unlike CASL[8], makes no attempt to maintain 'vorticity contours' and/or 'iso-areas'.

The code alternates between *discrete* and *continuous* forms of Eulerian fields (stream, velocity, vorticity, pressure et al), efficiently implemented in *Mathematica* 5 (interpolation/discretization). So all 'differentiations' are continuous, inverse Laplace – discrete. Basic steps:

For discrete $\zeta_d = \zeta_d(t)$ (on lattice)

1) Compute $\psi_d = \Delta^{-1}[\zeta_d]$ - via discrete convolution

2) Interpolate $\psi_d \rightarrow \psi(\boldsymbol{r})$,

3) Compute velocity: $\boldsymbol{v} = (-\psi_y, \psi_x)$,

pressure: $p_d = 2\Delta^{-1}[J(\psi_x|\psi_y)_d] \rightarrow p(\boldsymbol{r})$

acceleration: $\boldsymbol{a} = -\nabla p + \nu\Delta\boldsymbol{u} + \boldsymbol{F}$

4) Lagrange back-step: $\boldsymbol{r}_d' = \boldsymbol{r}_d - \boldsymbol{v}_d\,\mathrm{dt} + \boldsymbol{a}_d\,\frac{\mathrm{dt}^2}{2}$ (transformed lattice)

5) Evaluate $\zeta(\boldsymbol{r})$ on transformed lattice: $\zeta_d(t+\mathrm{dt}) = \zeta(\boldsymbol{r}_d', t)$

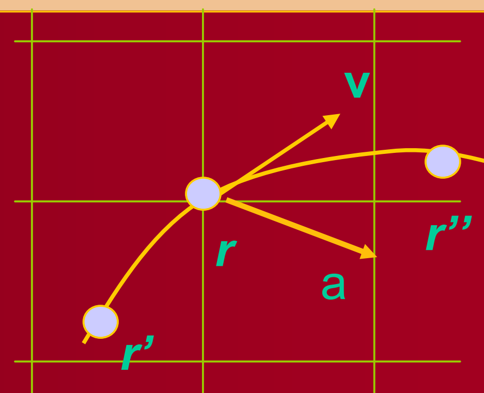


Fig. 4: *Lagrangian path with forward/ backward advected lattice point **r** =(x,y), using Lagrangian velocity **v** and acceleration **a**.*

The code creates effective '*local mixing*' due to interpolation and typically drives an arbitrary (random) initial vorticity to a 'dipole' or jet ('bar') equilibrium. We demonstrated it in ref. [4] for perturbed sn-sn. Many other examples of simulations are posted on web: http://www.cwru.edu/artsci/math/gurarie/fluid-gal.htm).

Dynamic and Arnold stability

So far we found only one dynamically stable soliton family: sn-sn dipole that relaxes to a close sn-sn "k-state" for all (even large) perturbations (Fig. 1). Other types maintain stability for a while, then undergo transformation to 'sn-sn' or 1D jet type.

For *Arnold stability* we use *invariant Lagrangian,* and its 2-nd variation – linear (Arnold) operator:

$$\mathscr{L} = -\frac{1}{2}\langle\zeta\,|\,\Delta^{-1}\,|\,\zeta\rangle + \iint F(\zeta)$$

$$L = \frac{\delta^2\mathscr{L}}{\delta\zeta^2} = (-\Delta)^{-1} + F'(\zeta) = (-\Delta)^{-1} + V(x,y)$$

For sinh-solitons potential function *V* is negative 'double well' (expressed as explicit algebraic function of θ). The condition for Arnold stability is negative-definite operator *L*

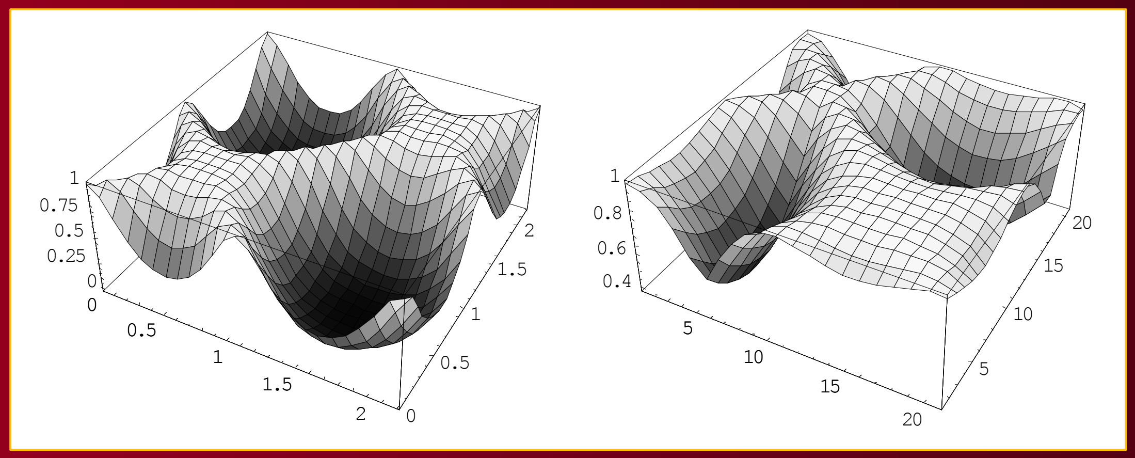
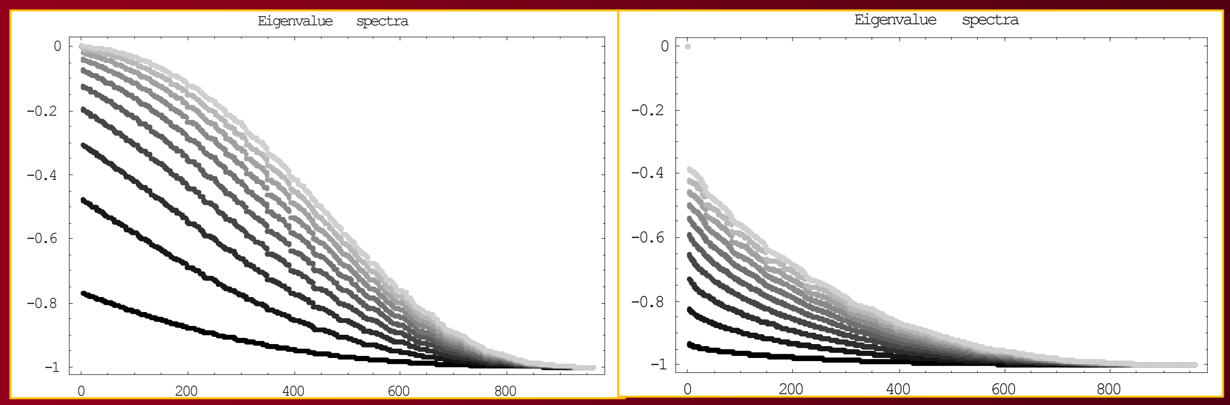


Fig. 5: *Potential functions V(x,y) of 'sn-sn' dipole (left) and 'sn-sn-dn' dipole (right).*

The eigenvalues spectra of operator $L=L_k$ are computed numerically (32x32 resolution) for the entire range of modulus: $0<k<3-\sqrt{8}$ ('sn-sn' – left plot, and 'sn-sn-dn' – right plot), different (increasing) curves correspond to increasing *k*.



Unlike *dynamic stability* (above), most soliton families prove *Arnold stable* for all *k*. The discrepancy is not surprising as two *stabilities* refer to different fluid systems: *ideal equilibria* vs. *quasi-inviscid dynamics*. The *dynamic relaxation* reveals, however peculiar feature of *Arnold spectra*: as unstable 'sn-sn-dn' type relaxes to its its 'sn-sn' terminal state, initial high *k* drops to a low (near linear) k_1 (for sn-sn), but the Arnold spectra (initial and terminal) look nearly identical.

SUMMARY AND REFERENCES

- Analytic ***Sinh-Poisson vortex solitons*** of an ideal or quasi-inviscid fluid gain insight into equilibration and ***stability*** of 2D flows.
- ***Arnold spectra*** could link two fluids (ideal and quasi-inviscid) and provide a new class of '***conserved integrals***' for quasi-inviscid flows.

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