Preface

It takes some chutzpah to write a linear algebra book. With so many choices already available, one must ask (and our friends and colleagues did): what is new here?

The most important context for the answer to that question is the intended audience. We wrote the book with our own students in mind; our linear algebra course is required for majors in mathematics, applied mathematics, and our joint degree in mathematics and physics, and it is the only linear algebra course most of them will take. Our course is also frequently taken by students in physics, computer science, and various fields of engineering. Linear algebra will be fundamental to most if not all of them, but they will meet it in different guises.

Most introductory linear algebra books fall into one of two categories: books written in the style of a freshman calculus text and aimed at teaching students to do computations with matrices and column vectors, or full-fledged "theorem–proof" style rigorous math texts, focusing on abstract vector spaces and linear maps, with little or no matrix computation. This book is different. We offer a unified treatment, building both the basics of computation and the abstract theory from the ground up, emphasizing the connections between the matrix-oriented viewpoint and abstract linear algebraic concepts whenever possible. The result serves students better, whether they are heading into theoretical mathematics or towards applications in science and engineering. Applied math students will learn Gaussian elimination and the matrix form of SVD, but they will also learn how abstract inner product space theory can tell them about expanding periodic functions in the Fourier basis. Students in theoretical mathematics will learn foundational results about vector spaces and linear maps, but they will also learn that Gaussian elimination can be a useful and elegant theoretical tool.

Another distinctive feature of this book is the organization of topics. Our perspective is that mathematicians invented vector spaces so that they could talk about linear maps; for this reason, we introduce linear maps as early as possible, immediately after the introduction of vector spaces. More generally, we have introduced topics we see as central (e.g., eigenvalues and eigenvectors) as early as we could, coming back to them again and again as we introduce new concepts which connect to these central ideas. This means that at the end of the course, rather than having just learned the definition of an eigenvector a few weeks ago, students will have worked with the concept extensively throughout the term, firmly (we hope) cementing it in their minds.

A perennial challenge in teaching linear algebra is that it is often the students' first foray into rigorous mathematics. Moving beyond the more problem-oriented calculus courses is a challenging transition, and we have done our best to facilitate it, actively working to help students build the mathematical maturity that they will need in this and future courses. The book is written in an accessible style; we have given careful thought to the motivation of new ideas and we have spent some time parsing difficult definitions and results after stating them formally. There are various pedagogical features aimed at helping the students learn to read a mathematics text: frequent "Quick Exercises" serve as checkpoints, with answers upside-down at the bottom of the page. Each section ends with a list of "Key Ideas", summarizing the main points of the section. Features called "Perspectives" at the end of some chapters collect the various different viewpoints on important concepts which have been developed throughout the text. The large selection of problems is a mix of the computational and the theoretical, the straightforward and the challenging.

The book begins with linear systems of equations over $\mathbb{R}$, solution by Gaussian elimination, and the introduction of the ideas of pivot variables and free variables. Section 1.3 discusses the geometry of $\mathbb{R}^n$ and geometric viewpoints on linear systems. We then move into definitions and examples of abstract fields and vector spaces.

Chapter 2 is on linear maps. They are introduced with many examples; the usual cohort of rotations, reflections, projections, and multiplication by matrices in $\mathbb{R}^n$, and more abstract examples like differential and integral operators on function spaces. Eigenvalues are first introduced in Section 2.1; the representation of arbitrary linear maps on $\mathbb{F}^n$ by matrices is proved in Section 2.2. Section 2.3 introduces matrix multiplication as the matrix representation of composition, with an immediate derivation of the usual formula. In Section 2.4, the range, kernel, and eigenspaces of a linear map are introduced. Finally, Section 2.5 introduces the Hamming code as an application of linear algebra over the field of two elements.

Chapter 3 introduces linear dependence and independence, bases, dimension, and the Rank-Nullity Theorem. Section 3.5 introduces coordinates with respect to arbitrary bases and the representation of maps between abstract vector spaces as matrices; Section 3.6 covers change of basis and introduces the idea of diagonalization and its connection to eigenvalues and eigenvectors. Chapter 3 concludes by showing that all matrices over algebraically closed fields can be triangularized.

Chapter 4 introduces general inner product spaces. It covers orthonormal bases and the Gram–Schmidt algorithm, orthogonal projection with applications to least squares and function approximation, normed spaces in general and the operator norm of linear maps and matrices in particular, isometries, and the QR decomposition.

Chapter 5 covers the singular value decomposition and the spectral theorem. We begin by proving the main theorem on the existence of SVD and the uniqueness of the singular values for linear maps, then specialize to the matrix factorization. There is a general introduction to adjoint maps and their properties, followed by the Spectral Theorem in the Hermitian and normal cases.

Finally, Chapter 6 is on determinants. We have taken the viewpoint that the determinant is best characterized as the unique alternating multilinear form on matrices taking value 1 at the identity; we derive many of its properties from that characterization. We introduce the Laplace expansion, give an algorithm for computing determinants via row operations, and prove the sum over permutations formula. The latter is presented as a nice example of the power of linear algebra: there is no long digression on combinatorics, but instead permutations are quickly identified with permutation matrices and concepts like the sign of a permutation arise naturally as familiar linear algebraic constructions. Section 6.3 introduces the characteristic polynomial and the Cayley–Hamilton Theorem, and Section 6.4 concludes the chapter with applications of the determinant to volume and Cramer's rule.

In terms of student prerequisites, one year of calculus is sufficient. While calculus is not needed for any of the main results, we do rely on it for some examples and exercises (which could nevertheless be omitted). We do not expect students to have taken a rigorous mathematics course before. The book is written assuming some basic background on sets, functions, and the concept of a proof; there is an appendix containing what is needed for the student's reference (or crash course).

Finally, some thanks are in order. To write a text book that works in the classroom, it helps to have a classroom to try it out in. We are grateful to the CWRU Math 307 students from Fall 2014, Spring and Fall 2015, and Spring 2016 for their roles as cheerful guinea pigs.

A spectacular feature of the internet age is the ability to get help type-setting a book from someone half-way around the world (where it may in fact be 2 in the morning). We thank the users of tex.stackexchange.com for generously and knowledgeably answering every question we came up with.

We began the project of writing this book while on sabbatical at the Institut de Mathématiques de Toulouse at the University of Toulouse, France. We thank the Institut for its warm hospitality and the Simons Foundation for providing sabbatical support (grant #267058 to E.M. and grant #264103 to M.M.). We also thank the National Science Foundation (grant DMS-1308725 to E. M.) and the Simons Foundation (grant #315593 to M. M.) for additional support.

And lastly, many thanks to Sarah Jarosz, whose album *Build Me Up From Bones* provided the soundtrack for the writing of this book