1. Suppose that $\|\cdot\|$ is a norm on $\mathbb{C}^n$ such that the maximum volume ellipsoid inside the unit ball of $\|\cdot\|$ is the Euclidean unit ball (so in particular, $\|v\| \leq |v|$ for all $v \in \mathbb{C}^n$). Show by the following steps that there is an orthonormal basis $(v_1, \ldots, v_n)$ of $\mathbb{C}^n$ such that $\|v_j\| \geq \frac{1}{2}$ for all $j \in \left\{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$.

   (a) Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be such that

   $$\det(T) = \max\left\{\det(S) \middle| S : \mathbb{C}^n \to \mathbb{C}^n, \max_{v \in \mathbb{C}^n, |v| \leq 1} \|Sv\| \leq 1\right\}.$$

   Show that for any $S : \mathbb{C}^n \to \mathbb{C}^n$,

   $$\det(T + \epsilon S) \leq \det(T)\|T + \epsilon S\|^n$$

   and that also

   $$\det(T + \epsilon S) = \det(T)(1 + \epsilon \operatorname{tr}(T^{-1}S) + o(\epsilon)).$$

   (b) Show that this implies that $\operatorname{tr}(T^{-1}S) \leq n \max_{v \in \mathbb{C}^n, |v| \leq 1} \|Sv\|$.

   (c) Let $P : \mathbb{C}^n \to V$ be orthogonal projection onto V, and let $S = TP$. Use the fact that $\dim(V) = \operatorname{tr}(P)$ to show that

   $$\max_{v \in \mathbb{C}^n, |v| \leq 1} \|TP\| \geq \frac{\dim(V)}{n}.$$

   (d) Choose the v_j inductively: let $|v_1| = 1$ such that $\|Tv_1\| = 1$. Use the above to choose $v_2 \in \langle v_1 \rangle^\perp$ so that $\|Tv_2\| \geq \left(\frac{n-1}{n}\right)$. Continue: show that this proves the claim.

2. Prove that if $Z_1, \ldots, Z_m$ are i.i.d. standard Gaussian random variables, then there is a constant so that

   $$\mathbb{E}[\max_{1 \leq j \leq m} |Z_j|] \geq c\sqrt{\log(m)}.$$