Math 423 Homework 6

1. Let $X = Y = [0, 1]$, let μ denote Lebesgue measure on X, and let ν denote counting measure on Y. Let $D = \{(x, x) : x \in [0, 1]\}$ denote the diagonal of $X \times Y$. Show that $\iint \chi_D d\mu d\nu$, $\iint \chi_D d\nu d\mu$ and $\int \chi_D d\mu \otimes \nu$ are all different.

2. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $f \geq 0$ be measurable. Define

$$G_f := \{x, y \in X \times [0, \infty) : y \leq f(x)\}.$$

   Show that $G_f \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$ and that $\mu \otimes m(G_f) = \int f d\mu$. This is the most general version of the statement "the integral of a nonnegative function is the area under the curve".