Math 423 Homework 3

1. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right-continuous, and let μ_F be the associated Lebesgue-Stieltjes measure. Show that F has left limits. For $a \in \mathbb{R}$, express $\mu_F(\{a\})$ in terms of F.

2. Let μ be a measure on $\mathcal{B}_\mathbb{R}$ such that

   (a) $\mu(E + x) = \mu(E)$ for all $x \in \mathbb{R}$ and all $E \in \mathcal{B}_\mathbb{R}$;

   (b) for some bounded interval I, $\mu(I) < \infty$.

   Prove that there is an $a > 0$ such that $\mu = am$, where m denotes Lebesgue measure on $\mathbb{R}$. That is, up to rescaling, Lebesgue measure is the unique translation-invariant Borel measure on $\mathbb{R}$.

3. Let $E \in \mathcal{L}$ with $m(E) > 0$. For any $\alpha \in (0, 1)$, show that there is an open interval I such that $m(E \cap I) > \alpha m(I)$.