Math 423 Homework 11

1. Show that the mean value theorem need not hold for F if F is only assumed to be absolutely continuous (hence a.e. differentiable).

2. Let X be a topological space and let $A \subseteq X$ be closed. Show that if $B \subseteq A$ is relatively closed, then in fact B is closed in X.

3. Let X be a non-empty set and let $\mathcal{F}$ be a collection of real-valued funtions on X. Show that the weak topology $\mathcal{T}$ generated by $\mathcal{F}$ is Hausdorff if and only if for all $x, y \in X$ with $x \neq y$, there is an $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

4. Let X be a topological space equipped with an equivalence relation. Let $\tilde{X}$ be the set of equivalence classes, and let $\pi : X \to \tilde{X}$ be the map (often called the projection map) which takes a point to its equivalence class. Let

$$\mathcal{T} = \{U \subseteq \tilde{X} : \pi^{-1}(U) \text{ is open in } X\}.$$

   (a) Show that $\mathcal{T}$ is a topology on $\tilde{X}$ (it's called the quotient topology).

   (b) Show that if Y is another topological space, then $f : \tilde{X} \to Y$ is continuous if and only if $f \circ \pi : X \to Y$ is continuous.

   (c) Show that $\tilde{X}$ is T_1 if and only if every equivalence class is closed.

   (d) Consider

$$\mathbb{SO}(2) = \left\{ \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}.$$

   Define an equivalence relation on $\mathbb{R}^2$ by $x \sim y$ if and only if $y = Ux$ for some $U \in \mathbb{SU}(2)$. Show that the set of equivalence classes together with the quotient topology is homeomorphic to $[0, \infty)$ (with the usual topology).