Name: Solutions Group:

Math 224 Quiz 4 – E. Meckes

1. Consider the system of differential equations

$$\frac{dx}{dt} = -\frac{1}{y}$$
$$\frac{dy}{dt} = -\frac{2}{x}.$$

(a) Show that the system is Hamiltonian and find the Hamiltonian function.

25 Note that $\frac{\partial}{\partial x}\left[\frac{dx}{dt}\right] = 0 = \frac{\partial}{\partial y}\left[\frac{dy}{dt}\right]$.

We're looking for $H(x,y)$ s.t.

$$\frac{\partial H}{\partial y} = -\frac{1}{y} \Rightarrow H(x,y) = -\log|y| + \phi(x)$$

$$\Rightarrow \frac{\partial H}{\partial x} = \phi'(x) \overset{!}{=} \frac{2}{x}$$

$$\Rightarrow \phi(x) = 2\log|x| + C$$ (but we don't bother with C.)

$\Rightarrow$ We can take

$$\boxed{H(x,y) = 2\log|x| - \log|y| = \log\left(\frac{x^2}{|y|}\right)}$$

as the Hamiltonian function.

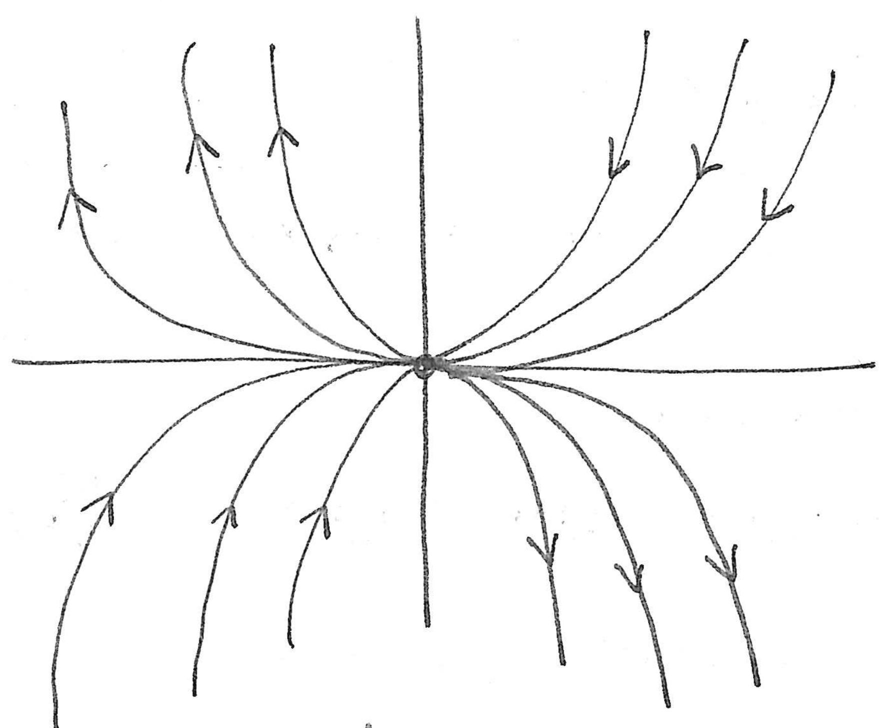
(b) Sketch the phase portrait of the system. Make sure to include directions on solution curves and include curves starting in each of the four quadrants. You will probably want to use the fact that $a\log(s) - \log(t) = \log\left(\frac{s^a}{t}\right)$.

20 We sketch level curves of H & add arrows to indicate the directions of solution curves in time.

Level curves have $\log\left(\frac{x^2}{|y|}\right) = c$

$$\Leftrightarrow \frac{x^2}{|y|} = \tilde{c} \Leftrightarrow |y| = \frac{1}{\tilde{c}} x^2.$$

So the level curves are parabolas through O:



Checking signs: $\frac{dx}{dt} > 0$ iff $y < 0$ and $\frac{dy}{dt} > 0$ iff $x < 0$.

This lets us add arrows as shown.

2. Consider the undamped harmonic oscillator $y'' + 4y = 0$.

(a) What is the natural period of this oscillator?

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The characteristic polynomial is

s^2+4 : roots are $\pm 2i$.

Solutions are therefore linear combinations of $\sin(2t)$ & $\cos(2t)$; the natural period is π.

(b) Find the solution of the initial value problem

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$$y'' + 4y = \sum_{n=1}^{\infty} \delta_{\pi n} \qquad y(0) = 0, y'(0) = 1.$$

Laplace transform both sides:

$$s^2\mathcal{L}[y](s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{1} + 4\mathcal{L}[y](s) = \sum_{n=1}^{\infty} e^{-\pi n s}$$

$$\mathcal{L}[y](s) = \frac{1}{s^2+4} + \sum_{n=1}^{\infty} \frac{e^{-n\pi s}}{s^2+4}$$

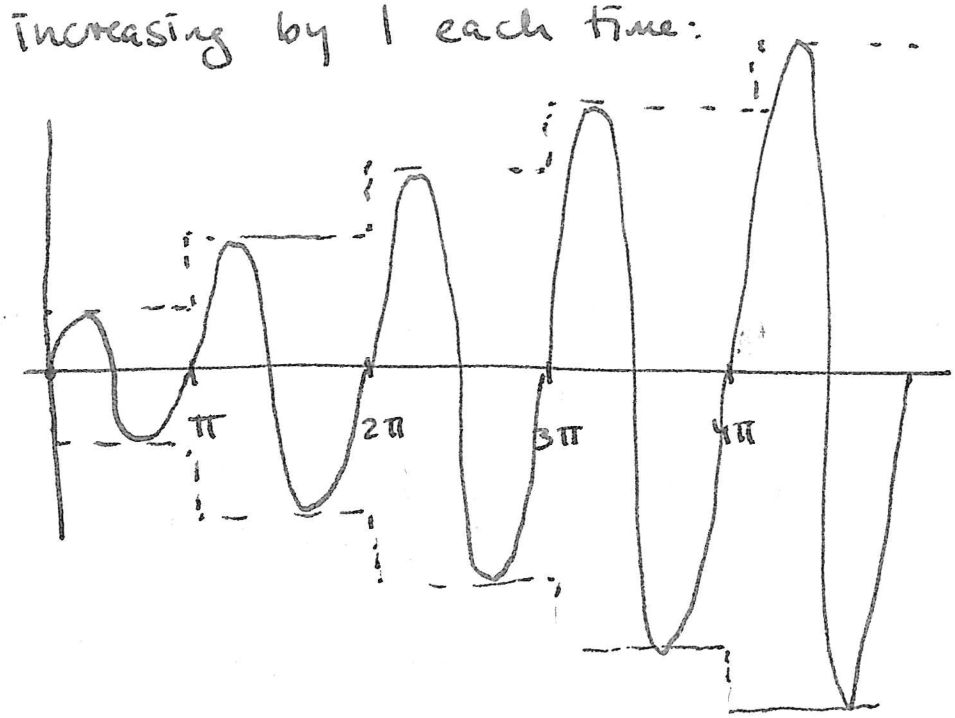
$$= \frac{1}{2}\mathcal{L}[\sin(2t)](s) + \sum_{n=1}^{\infty} \frac{1}{2} e^{-n\pi s}\mathcal{L}[\sin(2t)](s)$$

$$\Rightarrow \quad y(t) = \frac{1}{2}\sin(2t) + \sum_{n=1}^{\infty}\frac{1}{2}u_{n\pi}(t)\underbrace{\sin(2(t-n\pi))}_{\substack{=\sin(2t-2\pi n)\\ =\sin(2t)}}$$

$$= \frac{1}{2}\sin(2t)\left[1 + \sum_{n=1}^{\infty} u_{n\pi}(t)\right]$$

(c) Describe the long-term behavior of your solution in part (b). What is the connection between the timing of the delta functions (i.e., the choice of πn) and the natural period of the oscillator?

15 The term $1+\sum_{n=1}^{\infty} u_{n\pi}(t)$ is piecewise constant & adds 1 at each integer multiple of π, so we have a sine wave on each interval $[n\pi, (n+1)\pi]$ with amplitude increasing by 1 each time:



The oscillator oscillates at the natural frequency with ever increasing amplitude. This happens because the δ force is applied at the same point of each natural period.

3. Explain in words the idea behind the derivation of the improved Euler's method. You are welcome to include a picture as well. Your answer should be in complete, clear English sentences.

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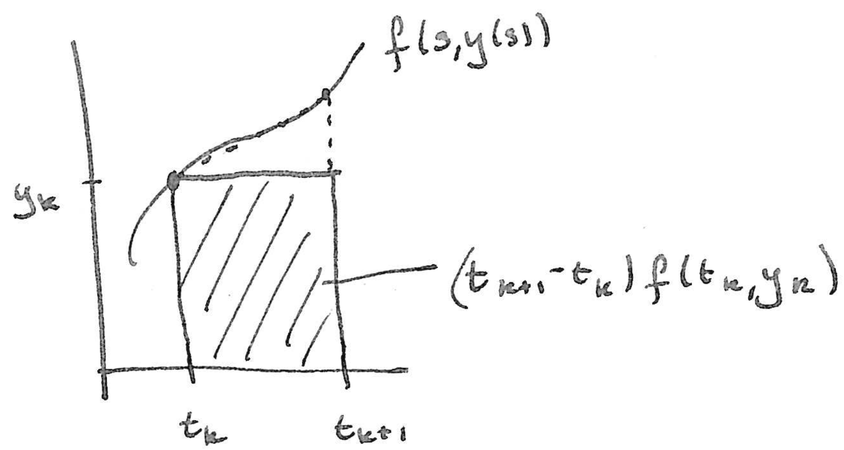
Hint: Recall that we started from

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(s, y(s))ds.$$

In Euler's method, we approximate

$\int_{t_k}^{t_{k+1}} f(s,y(s))\,ds$ by $(t_{k+1}-t_k)\, f(t_k, y_k)$,

Which corresponds to approximating area under the curve $(s, f(s,y(s)))$ by a rectangle:



Improved Euler's method uses the trapezoid rule instead: $\int_{t_k}^{t_{k+1}} f(s,y(s))ds \approx \frac{f(t_k, y_k) + f(t_{k+1}, y(t_{k+1}}{2}$.

And uses Euler's method to get a temporary approximation to $y(t_{k+1})$ in order to carry this out.

Hamiltonian systems: A two-dimensional system of differential equations is *Hamiltonian* if there is a function $H(x, y)$ such that

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}.$$

The function $H(x, y)$ is called the *Hamiltonian (function)* of the system.

Laplace transforms:

$$\mathcal{L}[y] = \int_0^\infty y(t)e^{-st}\,dt$$

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$$

$$\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0)$$

$y(t)$	$Y(s) = \mathcal{L}[y]$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$u_a(t)$	$\frac{e^{-sa}}{s}$
δ_a	e^{-as}
$u_a(t)f(t-a)$	$e^{-as}\mathcal{L}[f](s)$
$e^{at}f(t)$	$\mathcal{L}[f](s-a)$
$tf(t)$	$-\frac{d}{ds}\left(\mathcal{L}[f]\right)$

Numerical Methods

Euler: $y_{k+1} = y_k + \Delta t f(t_k, y_k)$

Improved Euler:

- $m_k = f(t_k, y_k)$
- $y^*_{k+1} = y_k + (\Delta t)m_k$
- $n_{k+1} = f(t_{k+1}, y^*_{k+1})$
- $y_{k+1} = y_k + (\Delta t)\left(\frac{m_k+n_{k+1}}{2}\right)$