Name: Solutions

Math 224 Quiz 2 – E. Meckes

1. Consider the model for the damped harmonic oscillator given by

$$y''(t) + 2y'(t) + 10y = 0.$$

(a) Show that $y_1(t) = e^{-t}\sin(3t)$ and $y_2(t) = e^{-t}\cos(3t)$ are both solutions to the differential equation above.

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$y_1' = -e^{-t}\sin(3t) + 3e^{-t}\cos(3t)$ 2 pts

$y_1'' = e^{-t}\sin(3t) - 3e^{-t}\cos(3t) - 3e^{-t}\cos(3t) - 9e^{-t}\sin(3t)$

$= -8e^{-t}\sin(3t) - 6e^{-t}\cos(3t)$ 4 pts

2 pts

$\Rightarrow y_1'' + 2y_1' + 10y_1 = e^{-t}\sin(3t)[-8 + 2(-1) + 10] + e^{-t}\cos(3t)[-6 + 2\cdot 3] = 0$

Similarly, $y_2' = -e^{-t}\cos(3t) - 3e^{-t}\sin(3t)$ 2 pts

4 pts

$y_2'' = e^{-t}\cos(3t) + 3e^{-t}\sin(3t) + 3e^{-t}\sin(3t) - 9e^{-t}\cos(3t)$

$= -8e^{-t}\cos(3t) + 6e^{-t}\sin(3t)$

2 pts

$\Rightarrow y_2'' + 2y_2' + 10y_2 = e^{-t}\cos(3t)[-8 + 2(-1) + 10] + e^{-t}\sin(3t)[6 + 2(-3)]$

$= 0.$

So both y_1, y_2 are solutions.

(b) Convert the second-order equation into a first-order system. What solutions to the system correspond to the solutions you were given to the second-order equation?

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$\frac{dy}{dt} = v$

$\frac{dv}{dt} = -10y - 2v$ 3 pts

$Y_1(t) = \begin{pmatrix} e^{-t}\sin(3t) \\ -e^{-t}\sin(3t) + 3e^{-t}\cos(3t) \end{pmatrix}$ $2\frac{1}{2}$ pts

$Y_2(t) = \begin{pmatrix} e^{-t}\cos(3t) \\ -e^{-t}\cos(3t) - 3e^{-t}\sin(3t) \end{pmatrix}$ $2\frac{1}{2}$ pts

(c) Give the general solution (to either the system or the second-order equation).

The general solution to the system is

8 $Y(t) = k_1 e^{-t} \begin{pmatrix} \sin(3t) \\ -\sin(3t)+3\cos(3t) \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} \cos(3t) \\ -\cos(3t) - \sin(3t) \end{pmatrix}$

8pts

(Note that at 0, $Y_1(0) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and $Y_2(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so

Y_1, Y_2 are linearly independent.)

The general solution to the second-order equation is therefore

8pts

$y(t) = k_1 e^{-t} \sin(3t) + k_2 e^{-t} \cos(3t).$

(d) Describe the typical long-term motion of the block in this model.

8 Unless the block starts at equilibrium ($y(0) = y'(0) = 0$), it will oscillate back & forth across the rest position, but get closer to rest (because of the e^{-t} factors) as time goes on. 8pts.

2. Recall the basic SIR Model of an epidemic:

$$\frac{dS}{dt} = -\alpha IS \qquad \frac{dI}{dt} = \alpha SI - \beta I \qquad \frac{dR}{dt} = \beta I,$$

where S is the portion of the population that is susceptible, I is the portion infected, and R is the portion "recovered"; i.e., not infected or susceptible.

Suppose now that the disease is evolving so that recovered people become susceptible to new strains at a rate proportional to the size of the recovered population.

(a) Modify the basic model to reflect this.

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$$\frac{dS}{dt} = -\alpha IS + \gamma R$$

$$\frac{dI}{dt} = \alpha SI - \beta I$$

$$\frac{dR}{dt} = \beta I - \gamma R$$

these terms show people moving from R to S at a rate γR (γ is an arbitrary parameter

(b) Give a two-dimensional version of the new model, involving only S and I. (Recall that $S(t) + I(t) + R(t) = 1$ for all t.)

$\rightarrow R = 1 - S - I$

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$$\frac{dS}{dt} = -\alpha IS + \gamma(1 - S - I)$$

$$\frac{dI}{dt} = \alpha SI - \beta I$$

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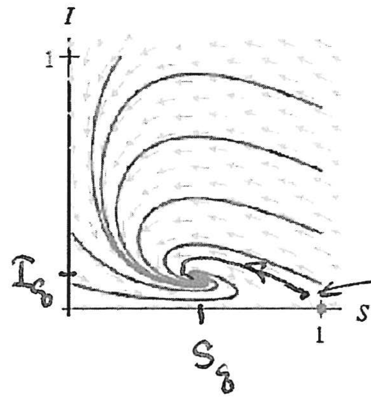
(c) Here is a picture of the phase plane of the two-dimensional model (for a particular choice of parameters). If the disease is initially introduced into the population by a small number of people, what happens in the long-term?



e.g., here: S_0 is almost 1 and I_0 is very small

Initially, the infected population rises, but then it starts to fall; the susceptible population falls (as more people become infected), then starts to rise again. The solution tends toward an equilibrium we can see at (S_8, I_8). This is a stable equilibrium (solutions are moving toward it) in which the disease is present ($I_8 \neq 0$), but under control.

3. Consider the linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{BY} = \begin{bmatrix} -2 & -3 \\ -3 & -2 \end{bmatrix} \mathbf{Y}$.

(a) Find the eigenvalues of B.

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The characteristic polynomial of B is

$\lambda^2 - \mathrm{tr}(B)\lambda + \det(B) = \lambda^2 + 4\lambda - 5$

$= (\lambda+5)(\lambda-1)$,

So the eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = 1$.

(b) Find the corresponding eigenvectors.

8

$\lambda_1 = -5$: We need solutions to $\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

We can take, e.g., $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda_2 = 1$: We need solutions to $\begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

We can take $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(c) Give the general solution to the system.

6 $Y(t) = k_1 e^{-5t}\begin{pmatrix}1\\1\end{pmatrix} + k_2 e^{t}\begin{pmatrix}1\\-1\end{pmatrix}$

(d) Solve the initial value problem $\dfrac{d\mathbf{Y}}{dt} = \begin{bmatrix}-2 & -3\\ -3 & -2\end{bmatrix}\mathbf{Y}$ and $\mathbf{Y}(0) = \begin{bmatrix}2\\0\end{bmatrix}$.

6 $\begin{pmatrix}2\\0\end{pmatrix} = k_1\begin{pmatrix}1\\1\end{pmatrix} + k_2\begin{pmatrix}1\\-1\end{pmatrix} \Rightarrow \begin{matrix}k_1 + k_2 = 1\\ k_1 - k_2 = 0\end{matrix}$

$\Rightarrow k_1 = k_2 = 1.$

$\Rightarrow Y(t) = e^{-5t}\begin{pmatrix}1\\1\end{pmatrix} + e^{t}\begin{pmatrix}1\\-1\end{pmatrix}$ solves the I.V.P.

(e) What is the long-term behavior of your solution as $t \to \infty$? What about $t \to -\infty$?

4 As $t \to \infty$, $x(t) = e^{-5t} + e^{t} \longrightarrow \infty$ and $y(t) = e^{-5t} - e^{t} \longrightarrow -\infty$. The parametric curve $(x(t), y(t))$ is asymptotic to the line in direction $\begin{pmatrix}1\\-1\end{pmatrix}$; i.e., the line $y = -x$.

As $t \to \infty$, $x(t)$ and $y(t)$ both approach $+\infty$. The parametric curve $(x(t), y(t))$ is asymptotic to the line in direction $\begin{pmatrix}1\\1\end{pmatrix}$; i.e., $y = x$.

(f) Sketch the phase plane for this system. Make sure to include any straight-line solutions, indicate direction of solution curves in time, and include the solution curve you found above to the initial value problem.

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