Name: Solutions Group Number:________

Math 224 Exam 1

September 16, 2019

1. (a) Solve the initial value problem

10 pts

$$\frac{dy}{dt} = \frac{1}{1-y}, \qquad y(0) = 0.$$

$\int (1-y)\,dy = \int dt \Rightarrow y - \frac{y^2}{2} = t + C$. To get $y(0) = 0$, we take $C = 0$: $y - \frac{y^2}{2} = t$.

Solving for y: $y^2 - 2y + 2t = 0 \Rightarrow y = \frac{2 \pm \sqrt{4-8t}}{2}$

$= 1 \pm \sqrt{1-2t}$.

Again, we need the initial condition to determine the sign. To get $y(0)=0$, we need $\boxed{y(t) = 1 - \sqrt{1-2t}}$.

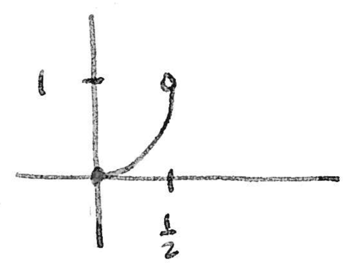
(b) What is the domain of definition of your solution in part (a)? Describe as precisely as possible what happens as t approaches the upper limit of the domain, and sketch your solution.

10 pts

The solution is defined for $t \in (-\infty, \frac{1}{2}]$.

As $t \nearrow \frac{1}{2}$, $y(t) \to 1$, (from below: $y(t) < 1$ for all $t < \frac{1}{2}$), and so

$\frac{dy}{dt} \to \infty$: the curve approaches height $y=1$ and is becoming vertical



2. Suppose a population of rabbits satisfies the logistic growth model:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right),$$

where k is the growth constant and N is the carrying capacity.

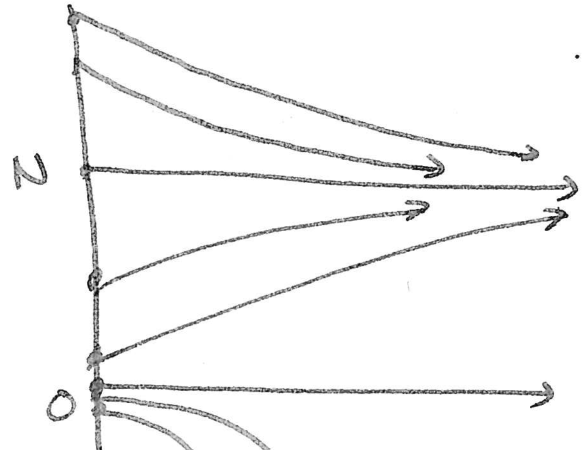
12 pts

(a) Sketch the phase line and classify the equilibria of the differential equation. Use the phase line to sketch representative solutions of the differential equation (all on the same axes).

Phase line

Equilibria are at P=0 and P=N. The equilibrium at P=0 is a source and the equilibrium at P=N is a sink.

Solutions:



(b) Now suppose that additional rabbits migrate into the area at a steady rate of α rabbits per unit time. Modify the equation to reflect this.

5 pts

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) + \alpha$$

(c) How do the equilibria of the system change based on the migration?

13 pts Equilibria occur at

$$0 = kP\left(1 - \frac{P}{N}\right) + \alpha$$

$$= -\frac{k}{N}P^2 + kP + \alpha$$

$\Rightarrow$ equilibria are at

$$P = \frac{-k \pm \sqrt{k^2 + \frac{4k\alpha}{N}}}{-\frac{2k}{N}}$$

$$= +\frac{N}{2} \pm \frac{N}{2}\sqrt{1 + \frac{4\alpha}{kN}}.$$

The first thing to notice is that 0 is no longer an equilibrium; that makes sense, since new rabbits keep arriving.

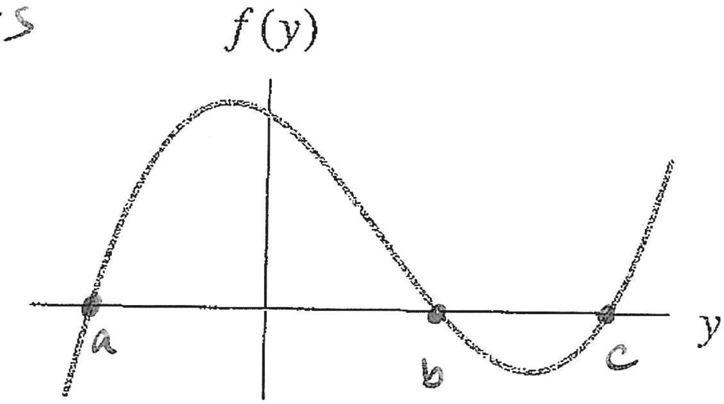
$\frac{N}{2} - \frac{N}{2}\sqrt{1 + \frac{4\alpha}{kN}} = \frac{N}{2}\left[1 - \sqrt{1 + \frac{4\alpha}{kN}}\right] < 0$, since $1 + \frac{4\alpha}{kN} > 1$.

So this equilibrium isn't interesting. Similarly,

$\frac{N}{2} + \frac{N}{2}\sqrt{1 + \frac{4\alpha}{kN}} = \frac{N}{2}\left[1 + \sqrt{1 + \frac{4\alpha}{kN}}\right] > N$. So the old equilibrium at N gets increased and the one at 0 goes away. The one above $\frac{N}{2}$ is still a sink ($\frac{dP}{dt} < 0$ if P is very large, and $\frac{dP}{dt} > 0$ if P is very small)

3. Suppose the following is a graph of the function $f(y)$.

20 pts

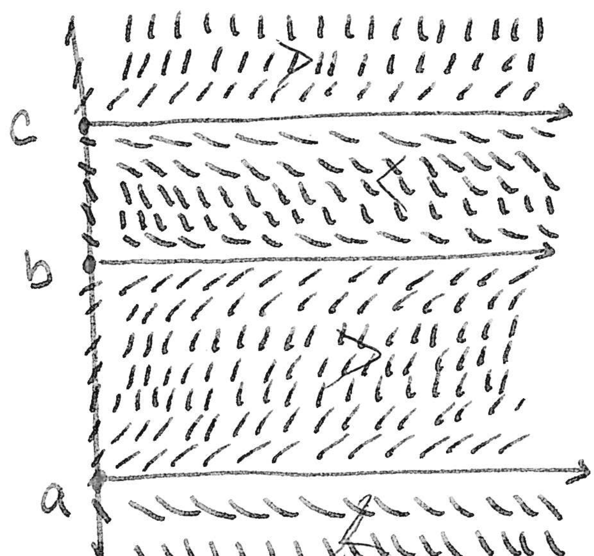


Sketch the slope field of the differential equation

$$\frac{dy}{dt} = f(y),$$

and describe the possible long term behaviors of the solutions, depending on their initial conditions.

(**Suggstion**: label important points on the axes of the graph above.)



Solutions with $y(0) = a, b, c$ are equilibria. If $y(0) > c$, solutions will increase to $+\infty$ (we don't know if they'll blow up in finite time) Solutions with $b < y(0) < c$ will decrease with $\lim_{t\to\infty} y(t) = b$ (but they won't reach b.) Solutions with $a < y(0) < b$ will increase with $\lim_{t\to\infty} y(t) = b$. Solutions with $y(0) < a$ will decrease to $-\infty$.

4. Consider the differential equation

$$\frac{dy}{dt} = -y + 2e^{-t}.$$

(a) Show that $y_1(t) = (2t-1)e^{-t}$ and $y_2(t) = (2t+1)e^{-t}$ are both solutions

10 pts

$$\frac{dy_1}{dt} = -(2t-1)e^{-t} + 2e^{-t} = -y_1(t) + 2e^{-t} \checkmark$$

$$\frac{dy_2}{dt} = -(2t+1)e^{-t} + 2e^{-t} = -y_2(t) + 2e^{-t} \checkmark$$

(b) What does the uniqueness theorem say about a solution with initial condition $y(0) = 0$?

5 pts

Since $y_1(0) = -1$ and $y_2(0) = 1$,

$y_1(0) < y(0) < y_2(0)$, and so, since

$-y + 2e^{-t}$ is continuous with continuous partial derivative w.r.t y, we will

have $y_1(t) < y(t) < y_2(t)$ for all t.

(c) Find the solution to the initial value problem

15 pts.

$$\frac{dy}{dt} = -y + 2e^{-t} \qquad y(0) = 0$$

and confirm that your claim in part (b) was true.

One option is to use part (a) together with the ~~linearity~~ linearity principle:

a solution to the homogeneous part is $y(t) = e^{-t}$, and from part (a), $y(t) = (2t-1)e^{-t}$ is a particular solution, so the general solution is

$$y(t) = (2t-1)e^{-t} + ke^{-t}.$$ To get $y(0)=0$, take $k=1$:

$$y(t) = (2t-1)e^{-t} + e^{-t} = 2te^{-t}.$$

You can also use variation of parameters from $y_h(t) = ke^{-t} \rightsquigarrow y(t) = k(t)e^{-t}$

$$\Rightarrow y'(t) = -y(t) + k'(t)e^{-t} = -y + 2e^{-t} \text{ if}$$

$$k'(t) = 2 \Rightarrow k(t) = 2t + C.$$

$\Rightarrow$ $y(t) = (2t+C)e^{-t}$ is the general solution, and to get $y(0) = 0$, we take $C=0$:

$y(t) = 2te^{-t}$ solves the IVP.