# Math 224 Final Exam

December 11, 2012

Name: ______________________________

Section (circle one):

| 9:30 | 10:30 (M. Meckes) | 10:30 (E. Meckes / Kotelenez) | 11:30 | 2:00 |
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|---|---|
| 1 | |
| 2 | |
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| 10 | |
| Total | |

No books, notes, or calculators.
You may remove the formula sheet at the back.

1. Two populations, one of predators and one of prey, are modeled by the system of differential equations

$$\frac{dx}{dt} = x(1-x) + xy,$$
$$\frac{dy}{dt} = 4y\left(1 - \frac{y}{2}\right) - 2xy.$$

(a) Which of $x(t)$ and $y(t)$ represents the predator population and which represents the prey population? Explain your answer.

(b) What would happen to each population in the absence of the other?

(c) Linearize the system about each nontrival equilibrium (that is, equilibria with $x > 0$ and $y > 0$) and classify the equilibria.

(d) Sketch the $x$- and $y$-nullclines of the system for $x \geq 0$ and $y \geq 0$, including arrows indicating the directions of solution curves which cross the nullclines.

(e) Suppose that $x(0) = 2$ and $y(0) = 1$. Predict the long-term fates of the populations.

2. Consider the forced undamped harmonic oscillator

$$\frac{d^2y}{dt^2} + 9y = 2\sin(3t)$$

with initial conditions $y(0) = y'(0) = 0$. Find the solution $y(t)$ and describe its long-term behavior.

3. Find the general solution of

$$\frac{dy}{dt} = \frac{\cos t}{y^2}.$$

4. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \qquad \text{where } \mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}.$$

(a) Find the trace, determinant, and eigenvalues of $\mathbf{A}$.

(b) Find eigenvectors corresponding to each eigenvalue.

(c) Find the general solution of the system.

(d) Find the particular solution with $\mathbf{Y}(0) = (1, 0)$.

(e) Sketch the phase portrait for the system, including arrows indicating the directions of solution curves.

5. Solve

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = \delta_2(t), \qquad y(0) = 1, \qquad y'(0) = 0.$$

6. Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} 2 & 1 \\ a & 0 \end{bmatrix} \mathbf{Y},$$

where $a$ is a parameter. Graph the corresponding curve in the trace-determinant plane and for each value of $a$ classify the type of equilibrium at the origin.

7. A certain population grows according to the logistic model

$$\frac{dP}{dt} = 4P\left(1 - \frac{P}{4}\right),$$

where $P$ is measured in thousands of individuals.

(a) Suppose that the population begins to be hunted at a rate of 3 thousand per unit time. Modify the above logistic equation to reflect this.

(b) Sketch the phase line for your modified equation and classify all the equilibria.

(c) What happens to the hunted population if it starts at size 2?

(d) What happens to the hunted population if it starts at size 1/2?

8. Find the general solution of

$$\frac{dy}{dt} = -2y + \sin t.$$

9. Consider the initial value problem

$$\frac{dy}{dt} = t - y^2, \qquad y(0) = 0.$$

(a) Estimate $y(2)$ using Euler's method with $n = 2$.

(b) Suppose you used a computer to implement improved Euler's method for the initial value problem above to estimate $y(2)$ using $n = 100$, and you believe your error to be approximately 0.005. How large should $n$ be so that you believe your error to be approximately 0.0002?

10. Find the general solution of

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \mathbf{Y}.$$

$$\mathcal{L}[y] = \int_0^\infty y(t)e^{-st}\ dt$$

$$\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$$

$$\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0)$$

| $y(t)$ | $Y(s) = \mathcal{L}[y]$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\sin\omega t$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos\omega t$ | $\frac{s}{s^2+\omega^2}$ |
| $u_a(t)$ | $\frac{e^{-sa}}{s}$ |
| $\delta_a$ | $e^{-as}$ |
| $u_a(t)f(t-a)$ | $e^{-as}F(s)$ |
| $e^{at}f(t)$ | $F(s-a)$ |
| $tf(t)$ | $-\frac{dY}{ds}$ |

**Euler's method:**

$$y_{k+1} = y_k + \Delta t f(y_k, t_k)$$

**Improved Euler's method:**

$$\tilde{y}_{k+1} = y_k + \Delta t f(y_k, t_k)$$

$$y_{k+1} = y_k + \frac{\Delta t}{2}\big(f(y_k, t_k) + f(\tilde{y}_{k+1}, t_{k+1})\big)$$

**Runge–Kutta method:**

$$m_k = f(y_k, t_k), \qquad \tilde{t}_k = t_k + \frac{\Delta t}{2}$$

$$\tilde{y}_k = y_k + \frac{\Delta t}{2} m_k, \qquad n_k = f(\tilde{y}_k, \tilde{t}_k)$$

$$\hat{y}_k = y_k + \frac{\Delta t}{2} n_k, \qquad q_k = f(\hat{y}_k, \tilde{t}_k)$$

$$\overline{y}_k = y_k + \Delta t q_k, \qquad p_k = f(\overline{y}_k, t_{k+1})$$

$$y_{k+1} = y_k + \frac{\Delta t}{6}\big(m_k + 2n_k + 2q_k + p_k\big)$$