Name: Solutions

Math 224 Exam 5

April 17, 2015

1. (a) Solve the initial value problem $\frac{d^2y}{dt^2} + 4y = 3\cos 2t$, $y(0) = y'(0) = 0$.

Homogeneous part: $y'' + 4y = 0 \Rightarrow y_h(t) = k_1\cos(2t) + k_2\sin(2t)$.

From the solution to the homogeneous part, we can see that the system is being forced at resonance. Complexifying gives

$$y_c'' + 4y_c = 3e^{2it},$$

and we want the real part of the particular solution here.

Because we're at resonance, we try $y_c(t) = \alpha t e^{2it} \Rightarrow y_c'(t) = \alpha e^{2it} + 2i\alpha t e^{2it}$

$$\Rightarrow y_c''(t) = 4i\alpha e^{2it} - 4\alpha t e^{2it}$$

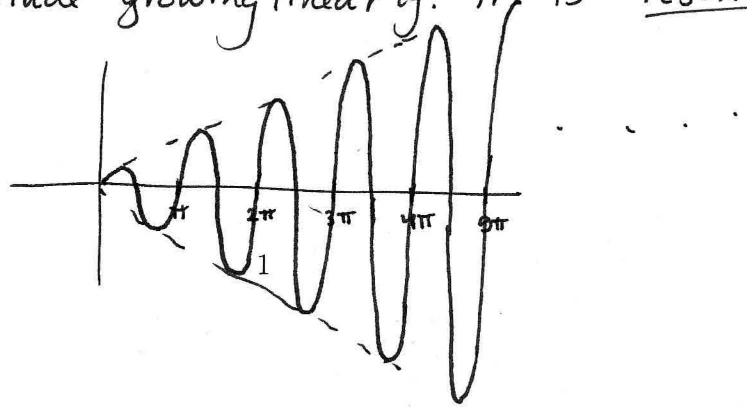
$$\Rightarrow y_c''(t) + 4y_c(t) = 4i\alpha e^{2it} \stackrel{?}{=} 3e^{2it} \Rightarrow \text{take } \alpha = \frac{3}{4i} = -\frac{3}{4}i$$

$$\Rightarrow y_p(t) = \mathrm{Re}[y_c(t)] = \mathrm{Re}\left[-\tfrac{3}{4}it\,e^{2it}\right] = \tfrac{3}{4}t\sin(2t)$$

$$\Rightarrow \boxed{y(t) = k_1\cos(2t) + k_2\sin(2t) + \tfrac{3}{4}t\sin(2t)}$$

(b) Describe the long-term behavior of the solution.

The long-term behavior of y is governed by the term $\frac{3}{4}t\sin(2t)$. This term oscillates with period π, with amplitude growing linearly: it is resonant!



2. Consider the system

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = -y - \sin x.$$

(a) Find the nullclines and equilibria.

x-nullcline: $\frac{dx}{dt} = y = 0 \Rightarrow y = 0$ (x-axis)

y-nullcline: $\frac{dy}{dt} = -y - \sin x = 0 \Rightarrow y = -\sin x$

Equilibria: when $y = 0$ and $\sin x = 0$

$\Rightarrow$ there are equilibria at $(n\pi, 0)$, where n is any integer.

(b) Determine the types of the equilibria.
Hint: There are two sets of equilibria of different types.

The Jacobian of the system is

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -\cos x & -1 \end{bmatrix}.$$

At an equilibrium of the form $(n\pi, 0)$ with n odd, we have $J = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$,

with char. poly $\lambda^2 + \lambda - 1$, and eigenvalues $\frac{-1 \pm \sqrt{5}}{2}$ (one positive, one negative)

so these equilibria are saddles.

At an equilibrium of the form $(n\pi, 0)$ with n even, $J = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$,

with char. poly. $\lambda^2 + \lambda + 1$ and eigenvalues $\frac{-1 \pm \sqrt{-3}}{2}$ (complex, negative real part),

so these equilibria are spiral sinks.

(c) Sketch the phase portrait for the system, including at least three equilibria.

- At $x = 2n\pi + \frac{\pi}{2}$, $y = 0$, we have $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = -\sin\left(\frac{\pi}{2}\right) = -1$ $\Rightarrow$ spirals at these equilibria are clockwise

- At $x = (2n+1)\pi$, $y = 0$, $J = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ with eigenvalues $\frac{-1 \pm \sqrt{5}}{2}$.

$\lambda = \frac{-1+\sqrt{5}}{2}$: $\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: eigenvector $\begin{pmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{pmatrix}$

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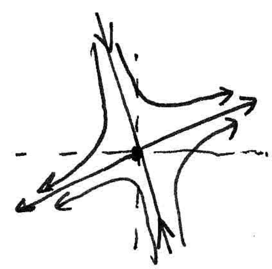
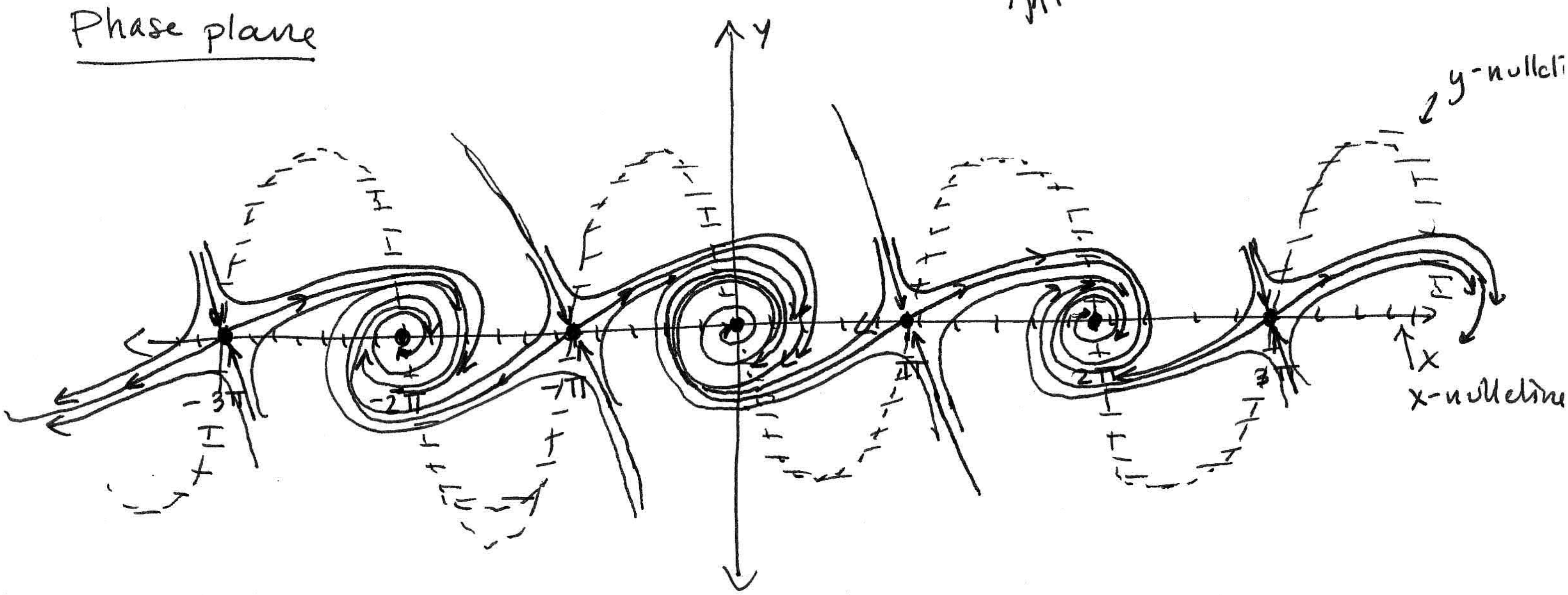
Local picture at these equilibria:



<u>Phase plane</u>



3. Consider the system

$$\frac{dx}{dt} = 2xy,$$
$$\frac{dy}{dt} = -y^2.$$

(a) Show that the system is Hamiltonian.

$$\frac{\partial}{\partial x}[2xy] = 2y$$

$$-\frac{\partial}{\partial y}[-y^2] = 2y.$$

Since these agree, the system is Hamiltonian.

(b) Find a Hamiltonian function $H(x, y)$ for the system.

$$\frac{\partial H}{\partial y} = 2xy \Rightarrow H(x,y) = \int 2xy \, dy = xy^2 + \phi(x)$$

$$\Rightarrow \frac{\partial H}{\partial x} = y^2 + \phi'(x) \stackrel{!}{=} y^2 \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c$$

$$\Rightarrow H(x,y) = xy^2 + c$$

Checking:

$$\frac{\partial H}{\partial y} = 2xy = \frac{dx}{dt}$$

$$\frac{\partial H}{\partial x} = y^2 = -\frac{dy}{dt} \quad \checkmark$$

4. Solve the initial value problem $\frac{dy}{dt} + 7y = u_2(t)$, $y(0) = 3$.

Taking the Laplace transform of both sides:

$$\mathcal{L}[y'](s) + 7\mathcal{L}[y](s) = \frac{e^{-2s}}{s}$$

$$\parallel$$

$$s\mathcal{L}[y](s) - y(0) + 7\mathcal{L}[y](s)$$

$$\parallel$$

$$(s+7)\mathcal{L}[y](s) - 3$$

$$\Rightarrow \quad \mathcal{L}[y](s) = \frac{3}{s+7} + \frac{e^{-2s}}{s(s+7)}.$$

$$\frac{1}{s(s+7)} = \frac{A}{s} + \frac{B}{s+7} \Rightarrow \frac{1}{s} = \frac{A(s+7)}{s} + B \overset{s=-7}{\Longrightarrow} B = -\frac{1}{7}$$

$$\Downarrow$$

$$\frac{1}{s+7} = A + \frac{Bs}{s+7} \overset{s=0}{\Longrightarrow} A = \frac{1}{7}$$

$$\Rightarrow \quad \mathcal{L}[y](s) = \frac{3}{s+7} + \frac{1}{7}\frac{e^{-2s}}{s} - \frac{1}{7}\frac{e^{-2s}}{s+7}$$

$$= 3\mathcal{L}[e^{-7t}](s) + \frac{1}{7}\mathcal{L}[u_2](s) - \frac{1}{7}e^{-2s}\mathcal{L}[e^{-7t}](s)$$

$$\Rightarrow \boxed{y(t) = 3e^{-7t} + \frac{1}{7}u_2(t) - \frac{1}{7}u_2(t)e^{-7(t-2)}}$$