Name: Solutions

Math 224 Exam 4

March 30, 2015

1. Consider the system $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$, where $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$.

(a) Find the eigenvalues of $\mathbf{A}$.

8 $Tr(A) = 2$, $Det(A) = -3 + 4 = 1 \Rightarrow$ Char. poly: $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$

$\Rightarrow$ There is one eigenvalue: $\boxed{\lambda = 1.}$

(b) Find the eigenvectors of $\mathbf{A}$.

8 $A - \lambda I = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$. Solving $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$;

take $\boxed{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$ as an eigenvector.

Any other eigenvector is a scalar multiple of this one.

(c) Find the general solution of the system.

8 We know that if A has a repeated ew and only one (up to scalar multiplication) ev, then the general solution is

$$Y(t) = e^t V_0 + te^t V_1, \quad \text{where} \quad V_1 = (A - \lambda I)V_0 = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} V_0.$$

(d) Find the solution of the system with the initial condition $\mathbf{Y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

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$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = Y(0) = V_0$. $\quad V_1 = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow \boxed{Y(t) = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

(e) Sketch the phase portrait, including the solution curve with the initial condition $\mathbf{Y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

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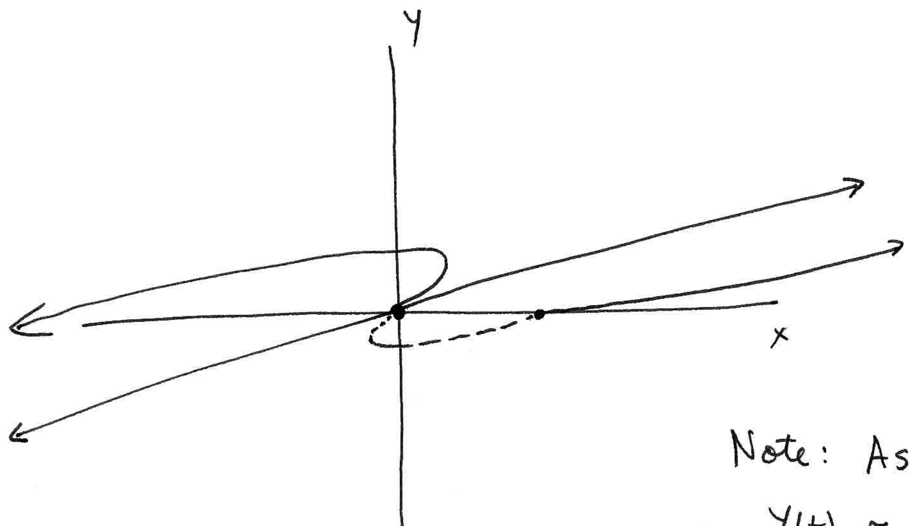
$Y'(0) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = ($



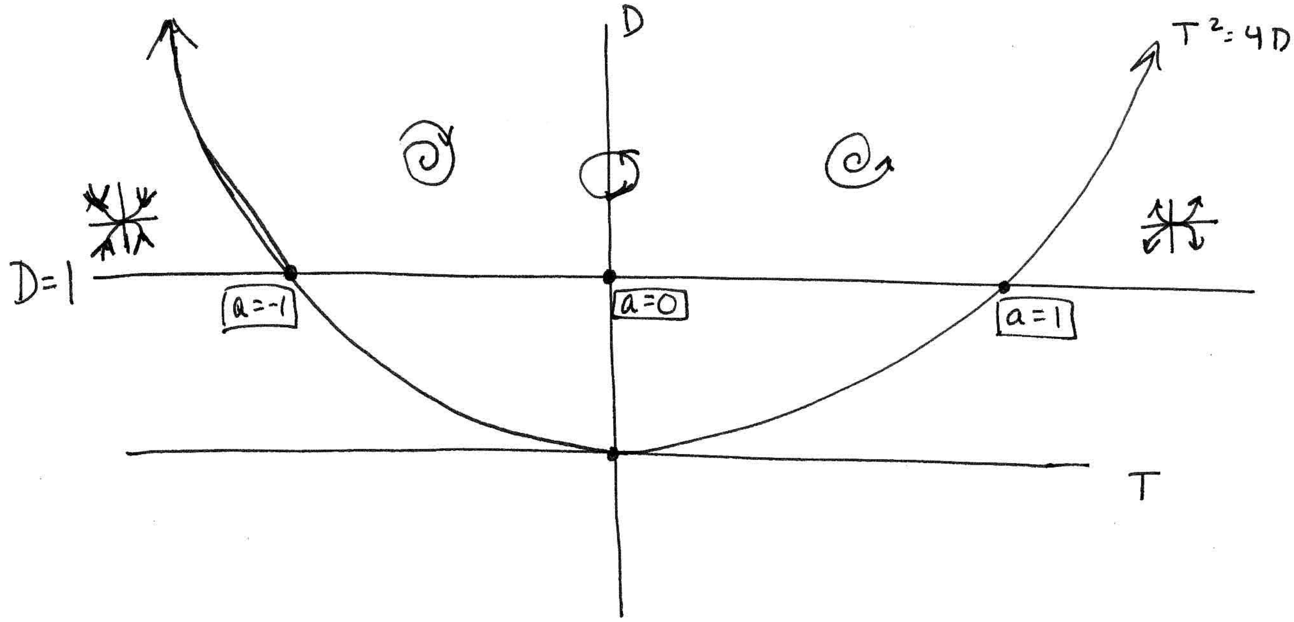
Note: As $t \to -\infty$,
$Y(t) \approx \underbrace{te^{-t}}_{\text{negative}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for $Y(t)$ as above; this shows why the solution curves around as $t \to -\infty$.

2. Consider the one-parameter family of linear systems given by

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & a+1 \\ a-1 & a \end{pmatrix} \mathbf{Y}.$$

(a) Sketch the corresponding curve in the trace-determinant plane.

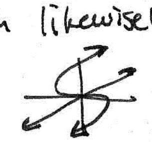
Trace $= 2a$ Determinant $= a^2 - (a+1)(a-1) = a^2 - [a^2-1] = 1$



(b) Identify which types of behaviors the system exhibits for which values of a.

Note that since $T = 2a$, the line $D=1$ intersects the D-axis when $a=0$. Moreover, $T^2 = 4D \iff 4a^2 = 4 \iff a = \pm 1$ and $T>0$ iff $a>0$, so the intersection points of $D=1$ with the parabola are at $a=\pm 1$, as labeled above.

- When $a<-1$, the system has a ~~source~~ sink at O.
- When $-1<a<0$, the system has a spiral sink at O.
- When $0<a<1$, the system has a spiral source at O.
- When $a>1$, the system has a source at O.
- When $a=0$ the system has a center at O.

- If $a=-1$, the matrix is $\begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix}$, which has only one eigenvector, so we have a pseudospiral sink: 
- If $a=1$, the matrix is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ which likewise has one eigenvector: we have a pseudospiral source 

3. Suppose a block with mass 1 is attached to the end of a spring with spring constant 5. The block is subject to a damping force proportional to its velocity, with a damping coefficient 4. Finally, an external time-dependent force of $\cos 2t$ acts on the block.

(a) Write a differential equation which models the behavior of the block.

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = \cos(2t)$$

(b) Find the general solution of your differential equation.

Homogeneous part: $y'' + 4y' + 5y = 0$, which has char. poly. $\lambda^2 + 4\lambda + 5$, so ews $\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

$\Rightarrow$ $y_{gen,h}(t) = k_1 e^{-2t}\cos(t) + k_2 e^{-2t}\sin(t)$.

For the particular solution, consider

$$y'' + 4y' + 5y = e^{2it}$$

and guess $y_c(t) = \alpha e^{2it}$ (so $y_c'(t) = 2i\alpha e^{2it}$, $y_c''(t) = -4\alpha e^{2it}$)

Need: $\alpha e^{2it}\left[-4 + 8i + 5\right] = e^{2it}$ $\Rightarrow$ $\alpha(1+8i) = 1$

$\Rightarrow$ $\alpha = \frac{1}{1+8i} = \frac{1-8i}{65}$

$\Rightarrow$ $y_c(t) = \left(\frac{1-8i}{65}\right)\left(\cos(2t) + i\sin(2t)\right) = \frac{1}{65}\left[\cos(2t) + 8\sin(2t)\right]$
$+ \frac{i}{65}\left[\sin(2t) - 8\cos(2t)\right]$

Since we are forcing with $\cos(2t)$, we need the real part,

$y_p(t) = \frac{1}{65}\left[\cos(2t) + 8\sin(2t)\right]$, and so

$$\boxed{y_{gen}(t) = k_1 e^{-2t}\cos(t) + k_2 e^{-2t}\sin(2t) + \frac{1}{65}\left(\cos(2t) + 8\sin(2t)\right)}$$

(c) Describe the long-term behavior of the block.

In the long term, the solution to the homogeneous equation dies out and there is steady-state oscillation (corresponding to y_p) with period π.

(d) Suppose that at time 0 the block is at rest and the spring is stretched so that the block is a distance 1 from its equilibrium position. Determine the position of the block for all times t.

$y(0) = 1, \quad y'(0) = 0$

$$y(t) = k_1 e^{-2t} \cos(t) + k_2 e^{-2t} \sin(t) + \tfrac{1}{65}(\cos(2t) + 8\sin(2t))$$

$$y'(t) = -2k_1 e^{-2t} \cos(t) - k_1 e^{-2t} \sin(t) - 2k_2 e^{-2t} \sin(t) + k_2 e^{-2t} \cos(t) + \tfrac{2}{65}(-\sin(2t) + 8\cos(2t))$$

$$\Rightarrow y(0) = k_1 + \tfrac{1}{65} \stackrel{?}{=} 1$$

$$y'(0) = -2k_1 + k_2 + \tfrac{16}{65} \stackrel{?}{=} 0$$

$$\Rightarrow k_1 = \frac{64}{65} \Rightarrow k_2 = 2k_1 - \frac{16}{65} = \frac{128-16}{65} = \frac{112}{65}$$

$$\Rightarrow \boxed{y(t) = \frac{64}{65} e^{-2t} \cos(t) + \frac{112}{65} e^{-2t} \sin(t) + \tfrac{1}{65}(\cos(2t) + 8\sin(2t))}$$