Name: Solutions

Math 224 Exam 2
February 18, 2015

1. Consider following two population models

$$① \quad \frac{dx}{dt} = 2x - 1.2xy, \quad \frac{dy}{dt} = -y + 0.9xy, \qquad\qquad \frac{dx}{dt} = 2x - 1.2xy, \quad \frac{dy}{dt} = y - 0.9xy. \quad ②$$

(a) One of the models is a predator/prey system, and the other models two competing species. Which is which (explain your answer)?

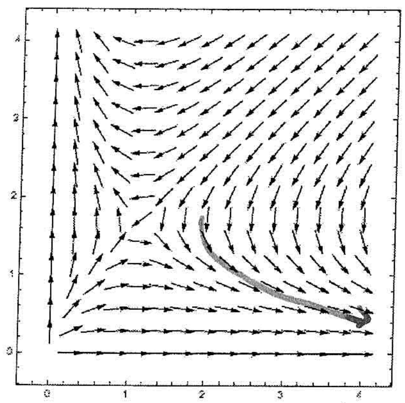
Model ① is predator/prey: the $-1.2xy$ term in $\frac{dx}{dt}$ means the presence of species y is bad for species x; the $.9xy$ term in $\frac{dy}{dt}$ means the presence of x is good for y.

Model ② is competing species: both xy terms are negative, so the presence of either species is bad for the other.

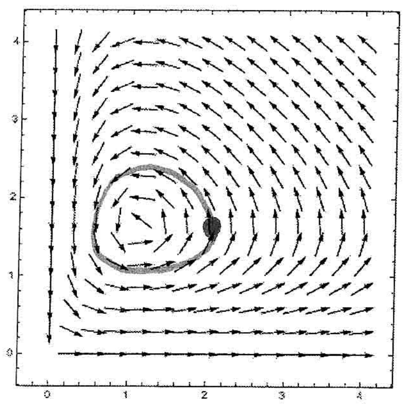
(b) For the predator/prey system, which variable represents the predators, and which represents the prey (explain)?

x is the prey (see above - the presence of y is bad for x) and y is the predator (the presence of x is good for y).

(c) Here are the direction fields for the two systems. Identify which direction field goes with which system.



②



①

Note: if x, y are both very large, then in system ①, $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} > 0$: ↖ . In ②, $\frac{dx}{dt} < 0$, $\frac{dy}{dt} < 0$: ↙

(d) For each system, sketch a solution curve on the direction field corresponding to the initial condition $\left(2, \frac{3}{2}\right)$. Describe the long-term behavior of the populations in both systems.

For system ① (on the right above), the two species coexist in a periodic cycle. For system ②, the y's die out and the x's grow (x out-competes y).

2. Give the general solution to

$$\frac{dx}{dt} = 3x + y,$$
$$\frac{dy}{dt} = -y.$$

$\frac{dy}{dt} = -y \Rightarrow y(t) = ke^{-t}$

$\Rightarrow \frac{dx}{dt} = 3x + ke^{-t}.$ (Non-homogeneous).

Solving the homogeneous part: $\frac{dx}{dt} = 3x \Rightarrow x(t) = \tilde{k}e^{3t}$

We guess a particular solution $x_p(t) = \alpha e^{-t}$.

Then $x_p'(t) = -\alpha e^{-t}$, and $3x_p(t) + ke^{-t} = (3\alpha + k)e^{-t}$

$\Rightarrow$ we need $-\alpha = 3\alpha + k \iff 0 = 4\alpha + k$

$\iff \alpha = -\frac{k}{4}$

By the theorem on linear ODEs, this means that the general solution for $\frac{dx}{dt} = 3x + ke^{-t}$ is $\tilde{k}e^{3t} - \frac{k}{4}e^{-t}$

$\Rightarrow$ General solution to the system: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \tilde{k}e^{3t} - \frac{k}{4}e^{-t} \\ ke^{-t} \end{pmatrix}$

$= \tilde{k}e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + ke^{-t}\begin{pmatrix} -\frac{1}{4} \\ 1 \end{pmatrix}$

3. Solve the initial value problem

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0, \qquad y(0) = 0, \quad y'(0) = 3.$$

We try a solution of the form $e^{st} = y(t)$:

$y'(t) = se^{st}$, $y''(t) = s^2 e^{st}$. So for y to be a solution, we need:

$e^{st}[s^2 + 7s + 10] = 0 \iff s^2 + 7s + 10 = 0$.

But $s^2 + 7s + 10 = (s+5)(s+2)$. So

$y_1(t) = e^{-2t}$ and $y_2(t) = e^{-5t}$ are both solutions. Unfortunately, neither satisfies the initial conditions: both have $y(0) = 1$, and $y_1'(0) = -2$, $y_2'(0) = -5$.

So we try combining them: $y(t) = k_1 e^{-2t} + k_2 e^{-5t}$.

~~Then $y'(t) = -2k_1 e^{-2t}$~~ y is still a solution:

$y'(t) = -2k_1 e^{-2t} - 5k_2 e^{-5t}$, $y''(t) = 4k_1 e^{-2t} + 25k_2 e^{-5t}$,

So $y'' + 7y' + 10y = e^{-2t}[4k_1 - 14k_1 + 10k_1] + e^{-5t}[25k_2 - 35k_2 + 10k_2] = 0$

Also, $y(0) = k_1 + k_2$ and $y'(0) = -2k_1 - 5k_2$.

Solving $\begin{array}{l} k_1 + k_2 = 0 \\ -2k_1 - 5k_2 = 3 \end{array} \Rightarrow -3k_2 = 3 \Rightarrow k_2 = -1 \Rightarrow k_1 = 1$.

So $\boxed{y(t) = e^{-2t} - e^{-5t}}$.

4. Usually in zombie movies, zombies do not stop infecting new victims until they are destroyed by a human; humans destroy as many zombies as they can. This leads us to the following variation of the SIR model (where H is the fraction of the initial population made of humans, Z is the fraction made of zombies, and $D = 1 - H - Z$ is the fraction of dead zombies, which we need not include explicitly):

$$\frac{dH}{dt} = -\alpha HZ,$$
$$\frac{dZ}{dt} = \alpha HZ - \gamma H.$$

(a) Calculate the equilibrium points of the model.

$\frac{dH}{dt} = 0 \Rightarrow \alpha HZ = 0 \Rightarrow \frac{dZ}{dt} = -\gamma H$

$\Rightarrow$ for $\frac{dH}{dt}$ and $\frac{dZ}{dt} = 0$, we need $H = 0$.

(And whenever $H = 0$, both $\frac{dH}{dt}$ and $\frac{dZ}{dt} = 0$):

Equilibrium solutions are

~~(0, t)~~ $(0, Z_0)$ for any $0 \le Z_0 \le 1$.

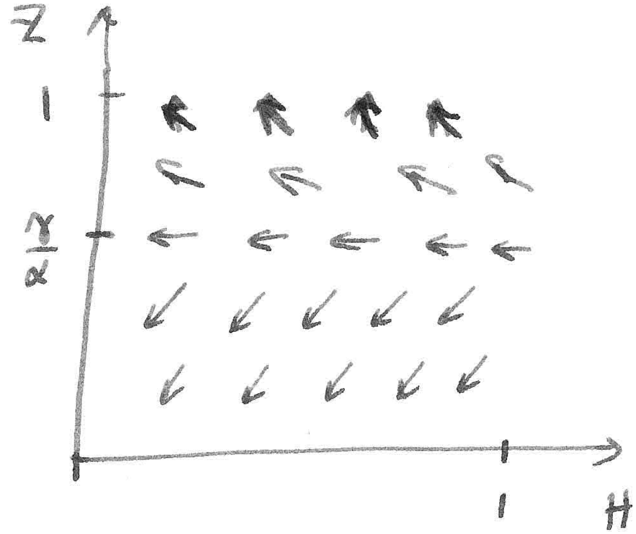
(b) Find the region of the phase plane where $\frac{dZ}{dt} > 0$.

For $\frac{dZ}{dt} = H(\alpha Z - \gamma) > 0$, we need

$\alpha Z - \gamma > 0 \Rightarrow$ ~~[illegible]~~ $Z > \frac{\gamma}{\alpha}$.

(We assume that $H, Z \geq 0$)

(c) Suppose that $\frac{\gamma}{\alpha} < 1$. Sketch the part of the phase portrait of the system where H and Z are positive. What does the model predict will happen to the human/zombie population?



If $Z = \frac{\gamma}{\alpha}$, then $\frac{dZ}{dt} = 0$ and $\frac{dH}{dt} < 0$.

In fact, $\frac{dH}{dt} < 0$ always.

We have $\frac{dZ}{dt} > 0$ if $Z > \gamma/\alpha$ and $\frac{dZ}{dt} < 0$ if $Z < \frac{\gamma}{\alpha}$

The model predicts that $\frac{\gamma}{\alpha}$ is a critical value for Z_0: if the initial zombie population is $\geq \gamma/\alpha$, all the humans die, but if the initial zombie population is $< \gamma/\alpha$, all the zombies are killed by humans.